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Working Papers | 088 | December 2010

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Measuring Inequality in CIS Countries: Theory and Empirics

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MEASURING INEQUALITY IN CIS COUNTRIES: THEORY AND EMPIRICS[1]

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Abstract

Distributions of many variables of interest in developed economic and financial markets, including income and wealth, exhibit heavy tails as in the case of Pareto or power laws. Many commonly used income and wealth inequality measures are very sensitive to extremes and outliers generated by these distributions due to their heavy-tailedness properties. This paper focuses on robust analysis of distributions and heavy-tailedness characteristics for data on income and wealth for the World, Russia and post-Soviet Central Asian economies. Among other results, it provides robust estimates of heavy-tailedness parameters for income and wealth in the markets considered and their comparisons with the benchmark values that are well-established for distributions of these variables in developed economies. The paper further provides applications of the obtained empirical results to inference on inequality measures and discusses their implications for market demand and economic equilibrium.

Keywords: Income inequality, wealth inequality, CIS countries, Russian economy, post-Soviet economies, heavy-tailedness, power laws, Pareto distribution, income inequality, market demand, economic equilibrium

JEL Classification: C14, D31, P24

[1] The authors gratefully acknowledge support from the Vienna Institute for International Economic Studies (wiiw) in the framework of the wiiw Global Development Network Southeast Europe and Commonwealth of Independent States (GDN SEE and CIS) Research Competition 2009. We also thank Mario Holzner, Ivica Urban and the participants at the GDN SEE and CIS Research Competition workshop, Vienna, Austria, June 2010.

1. INTRODUCTION

As discussed in numerous studies, economic growth and many other key economic variables, including consumer demand, are greatly affected by income and wealth inequality (see, among others, the review in Ibragimov and Ibragimov, 2007, Quadrini, 2008, and Appendix MD and references therein).

Empirical analyses on income inequality, poverty and market concentration and many other problems in economics and finance often face the difficulty that the data is heterogeneous or heavy-tailed in some unknown fashion. Heterogeneity and heavy-tailedness presents a challenge for applications of standard statistical and econometric methods. In particular, as pointed out by Granger and Orr (1972) and in a number of more recent studies (see, among others, Ch. 7 in Embrechts et al., 1997, Ibragimov, 2009, and references therein), many classical approaches to inference based on variances and (auto)correlations such as regression and spectral analysis, least squares methods and autoregressive models may not apply directly in the case of heavy-tailed observations with infinite second or higher moments.

More specifically, heavy-tailedness and heterogeneity complicate application of standard approaches to statistical inference and many commonly used measures of inequality, poverty and concentration (e.g., Cowell and Flachaire, 2007, Davidson and Flachaire, 2007, Gabaix, 2008, 2009, and the review in Ibragimov and Mueller, 2010). Some studies in the literature (e.g., Mandelbrot, 1997, Ch. E7), for instance, have criticized the use of one of often used measures of concentration, the Herfindahl-Hirschman index (HHI), by arguing that its distributional limits can be random. This holds whenever the firm sizes have heavy-tailed distributions that follow the empirically documented Zipf's laws (see Section 3). One can show that similar lack of consistency and non-Gaussian asymptotics under heavy-tailed observations also hold for a number of inequality and risk measures that have a structure similar to that of the HHI, such as the coefficient of variation and Sharpe ratio. In addition, the applicability of these measures becomes problematic under dependence and heterogeneity in the data generating process.

Several recent works in the literature have emphasized robustness as an important aspect in the choice of measures used in assessing economic inequality and estimation and inference methods for them (see, among others, Cowell and Flachaire, 2007, Davidson and Flachaire, 2007, and references therein).[2] The interest in robust inequality assessment is motivated, in part, by sensitivity of many income inequality measures to changes in different parts of the underlying income distribution, including sensitivity to extremes and outliers. The analysis of the effects of outliers on economic inequality measures is directly related to the study of heavy-tailedness phenomena and models for income and wealth distributions that exhibit heavy tails as in the case of commonly observed Pareto or power laws (see, for instance, Embrechts *et al.*, 1997, Gabaix, 2008, 2009, Ibragimov, 2009, and references therein). Heavy-tailed random variables (r.v.'s) $X>0$ with distribution that has power tails satisfy

$$P(X > x) \sim Cx^{-\zeta}, \tag{1}$$

as $x \to \infty$, with the tail index $\zeta>0$ (here and throughout the paper, $f(x) \sim g(x)$ means that $f(x) = g(x)(1 + o(1))$ as $x \to \infty$).[3] The tail index ζ characterizes the heaviness (the rate of decay) of the tails of power law distribution (1) (see Section 3). An important property of r.v.'s X satisfying a power law with the tail index ζ is that the moments of X are finite if and only if their order is less than ζ: $EX^p < \infty$ if and only if $p<\zeta$.

Empirical studies of income and wealth indicate that distributions of these variables in developed economies typically satisfy power laws (1) with the tail index ζ that varies between 1.5 and 3 for income and is rather stable, perhaps around 1.5, for wealth (see, among others, Gabaix, 2008, 2009, and references therein). This implies, in particular, that the mean is finite for income and wealth distributions (since $\zeta>1$). However, the variance is infinite for wealth (since $\zeta \approx 1.5<2$) and may be infinite for income (if $\zeta \leq 2$). In addition, since their tail indices are smaller than 3, income and wealth distributions have infinite third and higher moments.

The problem of infinite variance in wealth and income distributions is important because, as indicated before, it may invalidate or make problematic direct applicability of standard inference approaches, including regression analysis and least squares methods. On the other hand, the fact that the first moments of the distributions are finite is important and encouraging because

[2] See also Ibragimov (1997) and Ibragimov, Ibragimov and Sirajiddinov (2008) for some methods of indirect inference for income distributions and income inequality measures motivated by the related problems of missing data.

[3] More generally, one may require that $P(X > x) \sim Cl(x)\ x^{-\zeta}$, where $l(x)$ is a slowly varying function at infinity: $l(cx)/l(x) \to 1$, as $x \to \infty$, for all $c>0$. Most of properties and inference methods in the latter models are the same as in models (1).

it points to optimality of diversification and robustness of a number of economic models for the variables considered (see Ibragimov, 2009, and the discussion in Section 3).

As discussed in Section 3, many recent studies argue, using the data for developed economies, that the tail indices ζ typically lie in the interval (2, 4) for many financial returns and exchange rates. Among other results, for instance, Gabaix *et al.* (2003, 2006) present and discuss empirical estimates that support heavy-tailed distributions with tail indices $\zeta \approx 3$ for financial returns on many stocks and stock indices in developed markets. These results imply that, in contrast to income and wealth distribution, financial returns have finite variance (since $\zeta > 2$). Similar to the case of income and wealth distributions, financial returns have infinite fourth moments ($\zeta < 4$) and may have infinite moments of order 3 (if $\zeta \leq 3$).

2. RESEARCH OBJECTIVES

The discussion in the previous section illustrates that reliable inference on income and wealth distributions and their heavy-tailedness properties is crucial for estimation of inequality measures. In turn, robust inequality measurement is of great importance for the analysis of the effects of income and wealth disparity on economic markets and their development, including the changes in demand curves and the implied market equilibria over time (see the results and discussion in Ibragimov and Ibragimov, 2007, and Appendix MD).

Emerging economic markets are likely to be more volatile than their developed counterparts and subject to more extreme external and internal shocks. The higher degree of volatility suffered by these economies leads to the expectation that heavy-tailedness properties and distributions of key variables in these markets, including income and wealth, may be different from those in developed economies.

This paper focuses on robust analysis of distributions and heavy-tailedness characteristics for data on income and wealth for the World, Russia and post-Soviet Central Asian economies. Among other results, using the recently proposed robust tail index inference methods, the paper provides estimates of heavy-tailedness parameters ζ for income and wealth in the markets considered and their comparisons with the benchmark values $\zeta \in (1.5, 3)$ and $\zeta \approx 1.5$ that are well-established for distributions of these variables in developed economies (Sections 4-6).[4] The paper further provides applications of the obtained estimation results to inference on income inequality (Section 7). We also discuss applications of the empirical results in the analysis of the relation between inequality and consumer demand and their implications for economic equilibrium (Appendix MD).[5]

Our results point out to interesting and somewhat surprising similarities between the distributional characteristics and heavy-tailedness properties of income and wealth distributions in some of the economies considered and those in the developed markets. For instance, the estimates of the tail index ζ of income distribution in Russia are largely in agreement with the benchmark interval $\zeta \in (1.5, 3)$ for the income distribution in developed economies. This suggests that, apparently, the income distribution in Russia has achieved its equilibrium in terms of the likelihood of re-distributions and large fluctuations. Furthermore, the estimates indicate that the tail index is greater than 2 and, thus, the distribution has finite variance. Similar conclusions are obtained from the point estimates of the tail index of income distribution in Kazakhstan and from some of the results for Kyrgyzstan. At the same time, the estimates for Kyrgyzstan indicate that the income distribution in this country tends to be more heavy-tailed than in the case of the Kazakhstan and Russia.

Similar conclusions also hold for comparisons of the semiparametric estimates of the Gini coefficient G of inequality in the upper tails of the income distribution in the economies considered with the benchmark interval $G \in (0.2, 0.5)$ implied by the tail index estimates in the interval (1.5, 3) in developed markets.

The paper is organized as follows. Section 3 provides a review of the related literature on heavy-tailedness in economic and financial markets. Section 4 discusses inference methodology used in the analysis. Section 5 reviews the datasets used in the study and their sources. Section 6

[4] Related results recently obtained in Ibragimov, Ibragimov and Kattuman (2009) indicate that the tail indices for exchange rates in emerging markets differ from the values $\zeta \in (2, 4)$ in developed economies and tend to be smaller than the latter.

[5] Together with the results in Ibragimov and Ibragimov (2007) and Appendix MD, the estimates of heavy-tailedness parameters and inequality measures may also be used for inference on the volume of unofficial economy using the data on luxury goods consumption by households and estimates of elasticity of demand on luxuries (see the discussion in Section 8).

presents and discusses the main estimation results obtained in the paper. Section 7 discusses implications of the empirical results for income inequality measures. Section 8 makes some concluding remarks and reviews the suggestions for the further research. Appendices A and B contain the tables and diagrams on the empirical results obtained in the paper. Appendix WF provides a review of the Weber-Fechner law and the related size-rank regressions applied in the empirical analysis. Finally, Appendix MD provides new results on the relation between inequality and demand and discusses the implications of the empirical results on heavy-tailedness and inequality for market demand and economic equilibrium.

3. HEAVY TAILS IN ECONOMIC AND FINANCIAL MARKETS

The last four decades have witnessed rapid expansion of the study of heavy-tailedness phenomena in economic and financial markets. Following the pioneering work by Mandelbrot (1963) (see also Fama, 1965, and the papers in Mandelbrot, 1997), numerous studies have documented that time series encountered in many fields in economics and finance are typically heavy-tailed. In models involving a heavy-tailed positive r.v. X it is usually assumed that the distribution of X has power tails (1).

The parameter ζ in (1) is referred to as the tail index, or the tail exponent, of the distribution of X. It characterizes the degree of heavy-tailedness in power law (1) and the likelihood of occurrence of extreme observations and outliers in this distribution. As indicated in the introduction, for power moments of X one has: $EX^p < \infty$ if $p<\zeta$ and $EX^p = \infty$ if $p \geq \zeta$. In particular, the variables X that follow (1) with $\zeta \leq 2$ have infinite second moments: $EX^2 = \infty$. If (1) holds with $\zeta \leq 1$, then the first moment of X is infinite: $EX = \infty$. The following is a sample of estimates of the tail index ζ for (the absolute values of) returns on various stocks and stock indices: $3<\zeta<5$ (Jansen and de Vries, 1991), $2<\zeta<4$ (Loretan and Phillips, 1994), $\zeta \approx 3$ (Gabaix *et al.*, 2003, 2006). In the case $\zeta=1$, power law distributions (1) are commonly referred to as the Zipf's law. Zipf's law distributions with $\zeta=1$ have been found to hold for firm sizes (see Axtell, 2001, and Zhang, Chen and Wang, 2009) and city sizes (see Gabaix, 1999, for the discussion and explanations of the Zipf's law for cities).

Empirical results on power laws for income and wealth indicate that distributions of these variables in developed markets typically satisfy (1) with the tail index ζ that varies between 1.5

and 3 for income and is rather stable, perhaps around 1.5, for wealth (see Gabaix, 2008, 2009, and references therein).

As discussed in the introduction, the above tail index estimates imply, in particular, that the variances of financial returns in developed markets are finite; however, the returns typically have infinite fourth moments. In contrast, remarkably, wealth distributions have infinite variance, and the variance may also be infinite for income. Moreover, the distributions of income and wealth may even have infinite moments of order smaller than two. According to the estimates, the values $\zeta \in (1.5, 3)$ and $\zeta = 1.5$ are the critical boundaries between the orders of finite and infinite moments of income and wealth distributions in developed economies.

Besides the robustness properties of many empirical inequality, concentration and risk measures discussed in the introduction, heavy-tailedness, extremes and outliers may have dramatic effects on their population analogues, as in the case of the value at risk (VaR) analysis and the properties of a number of economic models (see Ibragimov, 2009, and references therein). In particular, as discussed in Ibragimov (2009) and Ibragimov, Jaffee and Walden (2009), diversification is typically preferable in the value at risk framework for moderately heavy-tailed risks with tail indices $\zeta > 1$. In contrast, diversification may increase portfolio VaR for extremely heavy-tailed risks with tail indices $\zeta < 1$ and infinite first moments.

The analysis of diversification for heavy-tailed variables directly relates to modeling and analysis of inequality using majorization relation and Lorenz curves (see, among others, Marshall and Olkin, 1979, the review in Ibragimov and Ibragimov, 2007, Appendix MD and references therein). The results in Ibragimov and Ibragimov (2007) and their extensions in Appendix MD provide sufficient conditions under which changes in income inequality lead to an increase or decrease in the market demand elasticities. The conditions are satisfied for individual demand functions commonly used in economic models, in particular, for the typical demand functions on luxury goods and necessities.

Importantly, the theoretical results in Ibragimov and Ibragimov (2007) help to explain, to some extent, the empirical results on consumer behavior reported in previous studies. The empirical study in Unnevehr and Khoju (1991) suggests that greater equality in income distribution reduces the average meat consumption. On the other hand, according to the empirical results in Pinstrup-Andersen and Caicedo (1978), reduction in income inequality has a considerable positive impact on the demand for food commodities, including meat. Senauer

(1990) reports that the lower-income households are more price responsive for the consumption of rice in developing countries. However, the analysis of the U.S. data on food commodities and household poverty status in Park, Hocomb, Raper and Capps (1996) provides estimates for the own-price elasticities that are similar between the income strata. Disparities in the above estimation results in the literature indicate that further empirical and theoretical analysis of the effects of income inequality on demand is highly desirable and provide further motivation for the analysis in the paper.

4. METHODOLOGY

Several approaches to inference about the tail index ζ of heavy-tailed distributions are available in the literature. The two most commonly used ones are Hill's estimator and the OLS approach using the log-log rank-size regression.

Let $X_1, X_2, \ldots, X_N>0$ be a sample from a population satisfying power law (1) (e.g., a sample of household income or wealth levels). Further, let, for $n<N$,

$$X_{(1)} \geq X_{(2)} \geq \ldots \geq X_{(n)} \geq X_{(n+1)} \quad (2)$$

be decreasingly ordered largest values of observations in the sample (that is, n+1 upper order statistics for the sample).

Hill's estimator $\hat{\zeta}_{Hill}$ of the tail index ζ is given by (see, among others, Embrechts *et al.*, 1997, Drees *et al.*, 2000, Gabaix, 2008, and references therein),

$$\hat{\zeta}_{Hill} = \frac{n}{\sum_{t=1}^{n} [log(X_{(t)}) - log(X_{(n+1)})]}. \quad (3)$$

The standard error of the estimator is $s.e._{Hill} = \frac{1}{\sqrt{n}} \hat{\zeta}_{Hill}$. The corresponding 95%-confidence interval for the true tail index ζ is thus given by

$$(\hat{\zeta}_{Hill} - \frac{1.96}{\sqrt{n}} \hat{\zeta}_{Hill}, \hat{\zeta}_{Hill} + \frac{1.96}{\sqrt{n}} \hat{\zeta}_{Hill}). \quad (4)$$

Hill's estimator may be simply motivated by the problem of estimating the parameter ζ for the Pareto distribution where (1) holds exactly for all values x greater than a certain threshold x_m:

$$P(X > x) = Cx^{-\zeta} \quad (5)$$

for all $x \geq x_m$, where $C = x_m^{\zeta}$. It is easy to see that, for a r.v. X satisfying (5), the log-transform $Y = log\ (X / x_m)$ follows an exponential distribution with the parameter ζ: $P(Y > y) = e^{-\zeta y}$ for all $y \geq 0$. Thus $\zeta = \frac{\mathbf{1}}{EY} = \frac{\mathbf{1}}{E\ log(\ X / x_m\ )}$. By the method of moments, this leads to estimation of the heavy-tailedeness parameter ζ, in the case where x_m is known, by the inverse of the sample mean $\overline{Y}_n = \frac{1}{n}\sum_{t=1}^{n} Y_t$ of the log-transforms $Y_t = log\ (X_t / x_m)$ for observations $X_1, X_2, ..., X_n$ from distribution (5):

$$\hat{\zeta} = \frac{n}{\overline{Y}_n} = \frac{n}{\sum_{t=1}^{n}[log(X_t) - log(x_m)]}. \qquad (6)$$

It is easy to see that $\hat{\zeta}$ is also the maximum likelihood estimator of ζ in (5). Similarly, the vector $(\hat{X}_m, \hat{\zeta})$, where $\hat{X}_m = \min_{t=1,...,n} X_t$ and

$$\hat{\zeta} = \frac{n}{\sum_{t=1}^{n}[log(X_t) - log\ (\hat{X}_m)]}, \qquad (7)$$

is the maximum likelihood estimator of (x_m, ζ) in the case of unknown x_m.

In addition to Hill's estimates of the tail indices of income and wealth distributions, we also provide tail index estimates obtained using robust modifications of log-log rank-size recently developed in Gabaix and Ibragimov (2011). These estimation procedures use the optimal shifts in ranks and the correct standard errors obtained in Gabaix and Ibragimov (2011).

It was reported in a number of studies that inference on the tail index using Hill's estimator suffers from several problems, including sensitivity to dependence in data and poor small sample properties (see Embrechts *et al.*, 1997, Ch. 6). Motivated by these problems, several studies have focused on the alternative approaches to the tail index estimation. For instance, Huisman *et al.* (2001) propose a weighted analogue of Hill's estimator that was reported to correct its small sample bias for sample sizes less than 1000. Embrechts *et al.* (1997), among others, advocated sophisticated non-linear procedures for tail index estimation.

Despite the availability of more sophisticated methods, a popular way to estimate the tail index ζ is still to run the following OLS log-log rank-size regression with $\lambda = 0$:

$$\log(t-\lambda) = a - b\cdot\log(X_{(t)}),\ t=1, ..., n, \tag{8}$$

or, in other words, calling t the rank of an observation, and $Z_{(t)}$ its size:

$$\log(Rank - \lambda) = a - b\cdot\log(Size) \tag{9}$$

(here and throughout the paper, $log(\cdot)$ stands for the natural logarithm). The reason for the popularity of the OLS approach to tail index estimation is arguably the simplicity and robustness of this method.

Regressions (8)-(9) with $\lambda = 0$ are motivated by the linear approximation $log\,[P(X > x)] \sim log\,(C) - \zeta\, log\,(x)$ implied by (1) and its empirical analogue $log\,(t/N) \approx log\,(C) - \zeta\, log\,(X_{(t)})$ for the observations $X_{(t)}$ in tails (2) of the distributions considered.[6]

In various frameworks, the log-log rank-size regressions of form (8)-(9) in the case $\lambda = 0$ and closely related procedures were employed, in particular, in Levy (2003), Levy and Levy (2003), Helpman *et al.* (2004), and many other works (see also the review and references in Gabaix and Ibragimov, 2011).

Unfortunately, the tail index estimation procedures based on OLS log-log rank-size regressions (8)-(9) with $\lambda = 0$ are strongly biased in small samples. The recent study by Gabaix and Ibragimov (2011) provides a simple practical remedy for this bias, and argues that, if one wants to use an OLS regression approach to tail index estimation, one should use the $Rank-1/2$, and run

$$\log(Rank - 1/2) = a - b\log(Size)\,, \tag{10}$$

that is,

$$\log(t - 1/2) = a - b\cdot\log(X_{(t)}),\ t=1, ...,n. \tag{11}$$

[6] In particular, similar to the estimates of Hill type (6)-(7), log-log regressions (8)-(9) with $n=N$ provide estimates of the heavy-tailedness parameter ζ in Pareto distributions (5) fitted to all income data available.

In (11), one takes the OLS estimate $\hat{b}$ as the log-log rank-size estimate $\widehat{\zeta}_{RS}$ of the tail index ζ. The shift of 1/2 is optimal, and reduces the bias to a leading order. The standard error of the estimator $\widehat{\zeta}_{RS}$ is $s.e._{RS} = \sqrt{\frac{2}{n}}\widehat{\zeta}_{RS}$ (the standard error is thus different from the OLS standard error given by $s.e._{RS} = \frac{1}{\sqrt{n}}\widehat{\zeta}_{RS}$). The corresponding correct 95% confidence interval for ζ is

$$\left(\widehat{\zeta}_{RS} - 1.96\times\sqrt{\frac{2}{n}}\widehat{\zeta}_{RS}\,;\ \widehat{\zeta}_{RS} + 1.96\times\sqrt{\frac{2}{n}}\widehat{\zeta}_{RS}\right). \qquad (12)$$

Numerical results in Gabaix and Ibragimov (2011) further demonstrate the advantage of the proposed approach over the standard OLS estimation procedures (8)-(9) with $\lambda = 0$ and indicate that it performs well under deviations from power laws and dependent heavy-tailed processes, including GARCH models. The modifications of the OLS log-log rank-size regressions with the optimal shift λ=1/2 and the correct standard errors provided by Gabaix and Ibragimov (2011) were subsequently used in Bosker *et al.* (2006), Bosker *et al.* (2008), Gabaix and Landier (2008), Hinloopen and van Marrewijk (2006), Ioannides *et al.* (2008), Zhang, Chen and Wang (2008) and several other works.

The paper further provides the estimates for several modifications of log-log rank-size regressions (8)-(9), such as log-linear size-rank regressions

$$\log(X_{(t)}) = a + b\cdot t,\ t=1, ..., N, \qquad (13)$$

that is,

$$\log\,(Size) = a + b\cdot\, Rank. \qquad (14)$$

Empirical log-linear size-rank relations (13)-(14) can be interpreted in terms of the Weber-Fechner law that this paper applies to income and wealth data for the first time in the literature in Appendix WF. In contrast to power laws and the implied log-log rank-size regressions (8)-(9) that hold for the *truncated* sample with n largest observations, log-linear size-rank regressions (13)-(14) and the corresponding Weber-Fechner laws discussed in Appendix WF are assumed to hold for the whole sample with N observations. Thus, regressions (13)-(14) and Weber-Fechner laws approximate the distribution of income over the whole population in contrast to log-log rank-size regressions of type (8)-(9) and power laws that hold only in the tails of income distribution.

We also provide estimation results for modifications of rank-size regressions (8)-(9) and (13)-(14) in the form of "hierarchy of logarithms" such as

$$log_m(X_{(t)}) = a + b \cdot t, \tag{15}$$

that is,

$$log_m(Size) = a + b \cdot Rank, \tag{16}$$

where $log_m(x) = \underbrace{log(log(log....log}_{m}(x)))$ is the m-th iteration of the logarithm.

The estimation results obtained using the methodology described in this section are presented and discussed in Section 6.

The estimates of the tail indices for income and wealth and the implied power law distributions (1) can be further used for semiparametric estimation of income and wealth inequality across countries in consideration. This estimation can be conducted using the expressions for the measures for underlying income and wealth distributions (see, for instance, Cowell and Flachaire, 2007, and Davidson and Flachaire, 2007). The discussion of these applications of the estimated income and wealth distributions and their heavy-tailedness parameters is presented in Section 7.

Among other results, the conclusions in Ibragimov and Ibragimov (2007) and their extensions discussed in Appendix MD indicate that an increase in income inequality decreases the demand elasticities for luxury goods and increases those for necessities. These conclusions allow one to use the estimates of income inequality, their changes over time and comparisons across countries for inference on consumer demand for different classes of goods. In particular, the empirical income inequality measures and elasticities for luxuries can be used to obtain estimates of demand for luxury goods and, indirectly, those of the volume of unofficial economy and its dynamics (see the discussion in Appendix MD and Section 8).

5. DATA

The datasets used in the empirical analysis in the paper are as described below.

The estimates for the World in the paper use the data on the worth of the wealthiest people of the planet in 2008 and 2009. The sample for 2008 contains the data for the owners of \$9 and more billions, and the sample for 2009 is for the owners of \$5 and more billions.

The dataset for Russia is from Rosstat (2007) and similar publications by the Rosstat for other years. Rosstat (the Federal State Statistics Service of Russian Federation) conducts sampling surveys of household budgets continuously during a calendar year in all subjects of Russian Federation. The surveys cover 48.7 thousands of households. The microdata on the survey results are provided by the Federal State Statistics Service online.[7]

The empirical results for Kyrgyzstan are mainly based on the sampling surveys conducted by the NSCKR, the National Statistical Committee of Kyrgyz Republic (see NSCKR, 2009, 2010*a*,*b*, and similar publications by the committee for other years). The yearly sampling surveys cover about 5,000 households and the NSCKR reports the results for a number of social and economic statistics, including those on measurement of standard of living and poverty in the country.[8] Similar surveys are also used in the empirical analysis for other post-Soviet Central Asian countries considered in the paper.[9]

For illustration, the diagrams in Appendix B.1 present the frequency distribution for the datasets under the analysis, and the diagrams in Appendix B.2 provide their cdf's. In addition, in Appendix B.2, we present the data on the dynamics of income inequality in Russia and the post-Soviet Central Asian countries needed for the discussion of the applications of the empirical results obtained to the inference on income inequality.

6. EMPIRICAL RESULTS

This section presents and reviews the estimation results obtained using the data on income and wealth distribution for the World, Russia, Kazakhstan, Kyrgyzstan, Tajikistan and Uzbekistan.

[7] http://www.micro-data.ru

[8] We also use the data on the frequency analysis of monthly earnings in Kyrgyzstan in 1994 conducted by the World Bank. The data is on the income levels of 4,489.3 thousands of households.

[9] For a review, see http://belstat.gov.by/homep/ru/links/links.php and

http://www.cisstat.com/rus/biblio-cis-list.htm

Tables A.1 provides some of the basic statistics for the datasets on income for the economies considered. Tables A.2 provide the tail index estimates $\hat{\zeta}_{RS}$ obtained using log-log rank-size regression (10)-(11) with the optimal shift λ= 1/2 and the correct standard errors $s.e._{RS} = \sqrt{\frac{2}{n}}\hat{\zeta}_{RS}$, as discussed in Section . The tables also provide the (correct) 95% confidence intervals (12) for the true tail indices ζ in (1) constructed using these standard errors. The last three columns of Tables A.2 also provide Hill's tail index estimates $\hat{\zeta}_{Hill}$, their standard errors $s.e._{Hill} = \frac{1}{\sqrt{n}}\hat{\zeta}_{Hill}$ and the corresponding 95% confidence intervals (4) for the tail indices ζ.

The inference results for Russia in Table A.2.Ru are presented for the number n of extreme observations (2) used in estimation equal to m%=10%, 5% and 1% of the total sample size N: $n=mN/100$. As indicated in the previous section, estimation for the World (Table A.2.W) is based on the dataset on n largest worth levels among all people and thus do not require truncation.

Due to the relatively small sizes of samples available for Kazakhstan, Kyrgyzstan, Tajikistan and Uzbekistan, the estimation results for these countries in Tables A.2.Kz, A.2.Kg, A.2.Tj and A.2.Uz are based on the tail truncation levels m=20%, 50% and 100%.[10]

For illustration, Diagrams A.2 following Tables A.2 provide the log-log rank-size plots that corresponds to the log-log rank-size regressions (10)-(11) estimated in the tables.

The estimation results for the worth distribution in the World in Table A.2.W are largely similar to the conclusions for developed countries in the literature that imply tail index estimates $\zeta \approx 1.5$ for wealth (see Sections 2 and 3). Namely, the confidence intervals constructed using Hill's estimates $\hat{\zeta}_{Hill}$ in Table A.2.W imply that the tail index ζ of the World worth distribution in 2008 lies in the interval $\zeta \in (1.4, 2.3)$ with 95% probability, and the tail index in 2009 satisfies $\zeta \in (1.3, 2.1)$ with 95% probability. Similarly, the confidence interval for ζ in 2009 constructed using the log-log rank-size regression estimate $\hat{\zeta}_{RS}$ implies $\zeta \in (1.4, 2.5)$ with 95% probability. The corresponding confidence interval for ζ in 2008 is (1.53, 2.69) and has the left-end point that is

[10] There is thus no truncation for m%=100% and $n=N$, and the inference results for this case correspond to the estimation of Pareto distributions (5) fitted to all the income data available.

very close to the value $\zeta \approx 1.5$. Importantly, the value $\zeta \approx 1.5$ lies in the 95% confidence intervals constructed using Hill's estimate for the worth data in 2008 and 2009 and in the 95% confidence interval obtained using the log-log rank-size regression estimate for the worth distribution in 2009. Thus, using these results, the null hypothesis H_0: $\zeta = 1.5$ is not rejected (in favor of the two-sided alternative H_a: $\zeta \neq 1.5$) at the 5% significance level for the worth distribution in 2008 and 2009. It is also important to note that Hill's point estimate $\hat{\zeta}_{Hill} = 1.69$ for the worth distribution in 2009 is close to the benchmark value $\zeta \approx 1.5$.

The results in Table A.2.Ru for the income distribution in Russia are largely in agreement with the empirical results on the tail indices $\zeta \in (1.5, 3)$ for income distribution in developed economies. Namely, all the log-log rank-size regression point estimates $\hat{\zeta}_{RS}$ and Hill's estimates $\hat{\zeta}_{Hill}$ in the table are very close to the value $\zeta = 3$. The most of these point estimates are slightly smaller than 3 and, thus, belong to the benchmark interval $\in (1.5, 3)$. Similarly, the most of the confidence intervals constructed using the estimates $\hat{\zeta}_{RS}$ and $\hat{\zeta}_{Hill}$ in the table either lie in the interval (1.5, 3) or have their larger parts lying in the interval. Furthermore, this is the case for the tail index estimates and the corresponding confidence intervals constructed using different tail truncation levels m% (=10%, 5% and 1%). For instance, according to the confidence intervals in the last three rows of Table A.2.Ru constructed using the estimates $\hat{\zeta}_{RS}$ and $\hat{\zeta}_{Hill}$ for different truncation levels m%, the tail index ζ of the income distribution in Russian in the 4th quarter of 2007 satisfies $\zeta \in (2.3, 3.2)$ with 95% probability. Similar conclusions also hold for other time periods in the table.

Importantly, the left end-points of *all* the confidence intervals in Table A2.Ru are greater than 2. That is, the null hypothesis H_0: $\zeta = 2$ is rejected in favor of H_a: $\zeta > 2$ at the 2.5% significance level for all the time periods dealt with. These conclusions thus imply that the variance of the income distribution in Russia is finite.

Similar to the point estimates $\hat{\zeta}$, the right-end points of all the confidence intervals in Table A2.Ru are close to the right boundary (=3) of the interval $\zeta \in (1.5, 3)$ in developed markets. Thus, similar to the case of developed markets (see the discussion in Sections 1 and 3), the third moment is very likely to be infinite for the income distribution in Russia.

In addition, the right-end points of all the confidence intervals are smaller than 4. This implies that the null hypothesis H_0: $\zeta=4$ is rejected in favor of H_a: $\zeta<4$ at the 2.5% significance level in all of the time periods in the table. Consequently, similar to the developed economies, the income distributions in Russia has infinite fourth moment.

The qualitative agreement of the results in Table A.2.Ru with those for developed economies in the literature suggests that, apparently, the income distribution in Russia has already reached its equilibrium in terms of the likelihood of extreme fluctuations and re-distributions.

As indicated before, the estimation results for the income distribution in Central Asian economies in the rest of Tables A.2 are based on rather small samples. Due to small sample sizes, the standard errors of the tail index estimates in the tables are quite large and the corresponding confidence intervals are rather wide. In addition, as discussed above, the small sample sizes require one to take the tail truncation levels m% to be rather large (e.g., equal to 20% or 50%, and also fit power laws (1) and Pareto distributions (5) to the whole sample of observations available with m%=100% and $n=N$) in order to increase the number of the largest order statistics (2) used in the tail index estimation.

Importantly, similar to the results for Russia discussed above, all of the log-log rank-size regression point estimates $\widehat{\zeta}_{RS}$ of the tail index ζ for the income distribution in Kazakhstan in Table A.2.Kz either lie in the interval $\zeta\in(1.5, 3)$ or are very close to its right boundary of 3. Similarly, Hill's tail index point estimates $\widehat{\zeta}_{Hill}$ for Kazakhstan are close to 3 as well. This suggest that the heavy-tailedness properties of the income distribution in Kazakhstan are similar to those in Russia and the developed countries with $\zeta\in(1.5, 3)$.

Similarly, the majority of the log-log rank-size regression point estimates $\widehat{\zeta}_{RS}$ of the tail index of the income distribution in Kyrgyzstan in Table A.2.Kg lie in the interval (1.5, 3), and the rest of the estimates are close to its left boundary of 1.5. In addition, about a half of Hill's point estimates $\widehat{\zeta}_{Hill}$ in the table either lie in the above interval or are close to the right boundary 3. These results suggest similarities of the heavy-tailedness properties of the income distribution in Kyrgyzstan to those in the developed markets, Russia and Kazakhstan. However, according to the point estimates, the income distribution in Kyrgyzstan tends to be more heavy-tailed than in the case of the latter countries. Namely, the tail index for the income distribution in Kyrgyzstan is likely to be smaller than in the case of Kazakhstan and Russia.

Unfortunately, the samples sizes for Tajikistan and Uzbekistan in Tables A.2.Tj and A.2.Uz are very small. A number of the log-log rank-size regression and Hill's point tail index estimates in the tables lie in the interval (1.5, 3) or are close to its boundaries. However, several of the point estimates are rather distant from the interval. The problem of robust tail index estimation for Tajikistan and Uzbekistan and other Central Asian economies needs to be revisited using large datasets and is left for further research (see also the discussion in Section 8).

Tables A.3 present the estimation results for the Weber-Fechner laws in the form of linear-log size-rank regressions (13)-(14) for the World, Russia and Kazakhstan (see Section 4 and Appendix WF). Diagrams A.3 illustrate the Weber-Fechner laws in form (WF.1) for the economies under the analysis. According to the results in Tables A.3, the linear-log size-rank regressions and Weber-Fechner laws provide remarkably good approximations to the worth distribution among the wealthiest people in the World and to income distributions among the *whole* population in Russia and Kazakhstan. This indicates that Weber-Fechner laws provide convenient approaches to modeling of wealth and income distribution among *all* households in a population. Such approaches may be used to complement power law analysis applied to extreme observations on income levels.

Table A.4 provides estimation results for the modification of log-log rank-size regression in the form of hierarchy of logarithms (15)-(16) applied to the data on the level of worth of the wealthiest people in the World. The results in Table A.4 imply the approximations to the distribution of the worth in the data using exponent iterations provided in the note under the table.

7. APPLICATIONS TO INEQUALITY AND POVERTY MEASUREMENT

Empirical analysis of heavy-tailedness parameters and distributions of income and wealth is important, in part, because it can be used in semiparametric estimation of income inequality and poverty measures. This estimation can be conducted using the expressions for the measures for underlying income and wealth distributions (see, for instance, Cowell and Flachaire, 2007, and Davidson and Flachaire, 2007).

This section discusses applications of the empirical analysis of tail indices and heavy-tailedness properties of income distributions in semiparametric inference on inequality and poverty measures. As an illustration, we focus on the analysis of the most commonly used measure of the inequality, the Gini coefficient. In complete similarity, one can obtain analogous estimates for other inequality measures.

It is well-known that, for Pareto income distribution (5), the Lorenz curve is given by $L(y) = 1-(1-y)^{1-1/\zeta}$, and the corresponding Gini coefficient is $G = G(\zeta) = \frac{1}{2\zeta - 1}$. It is important to note that $G(\zeta)$ is decreasing in ζ. Therefore, a as expected, a higher degree of heavy-tailedness in the underlying income distribution and the implied greater likelihood of occurrence of extremes and outliers in it (that is, a smaller value of the tail index ζ) translates into greater inequality as measured by the Gini coefficient *G.* Also, the benchmark interval $\zeta \in (1.5, 3)$ for the tail indices ζ of income distributions in the developed economies corresponds, in the Pareto case, to the benchmark values $G = G(\zeta) \in (0.2, 0.5)$ for their Gini coefficients. Similarly, the benchmark value $\zeta = 1.5$ for the wealth distributions in the developed markets corresponds to the benchmark value $G = G(\zeta) = 0.5$ of the Gini coefficient.

Naturally, in the case of a good fit of Pareto model (5) to the whole underlying income distribution, the estimates $\hat{\zeta}$ of the heavy-tailedness parameter ζ (e.g., estimates (6)-(7) and those obtained using log-log rank-size regressions (8)-(9) with *n*=*N*) imply the corresponding estimates $G = G(\hat{\zeta}) = \frac{1}{2\hat{\zeta} - 1}$ of the Gini coefficient of income inequality.

Similar to the Pareto case, the estimates $\hat{\zeta}$ of the tail index ζ in power law models (1) can be used to obtain the corresponding estimates of the inequality measure *G* in the upper tails (e.g., upper 10%, 5% and 1%) of the income distributions considered. That is, the semiparametric estimates of *G* in the upper tails are given by $\hat{G} = G(\hat{\zeta}) = \frac{1}{2\hat{\zeta} - 1}$. Using the delta method, we obtain that the standard error of $\hat{G} = G(\hat{\zeta})$ equals to $s.e._{\hat{G}} = G'(\hat{\zeta}) \cdot s.e._{\hat{\zeta}} = \frac{2 \cdot s.e._{\hat{\zeta}}}{(2\hat{\zeta} - 1)^2}$, where $s.e._{\hat{\zeta}}$ is the standard error of the estimator $\hat{\zeta}$.

For instance, for the log-log rank-size estimator $\widehat{\zeta}_{RS}$ of the tail index ζ, the standard error of $\widehat{G}_{RS} = G(\hat{\zeta}_{RS})$ is $s.e._{\widehat{G}_{RS}} = \frac{2}{(2\hat{\zeta}_{RS}-1)^2} \cdot \sqrt{\frac{2}{n}}\hat{\zeta}_{RS}$. The standard error of $\widehat{G}_{Hill} = G(\hat{\zeta}_{Hill})$ for Hill's estimator $\widehat{\zeta}_{Hill}$ is $s.e._{\widehat{G}_{Hill}} = \frac{2}{(2\hat{\zeta}_{Hill}-1)^2} \cdot \sqrt{\frac{1}{n}}\hat{\zeta}_{Hill}$.

As usual, the corresponding 95% confidence intervals for the value G in the upper tails of income distributions considered are given by $(\widehat{G} - 1.96 \cdot s.e._{\widehat{G}}, \widehat{G} + 1.96 \cdot s.e._{\widehat{G}})$, so that the 95%-confidence intervals constructed using the log-log rank-size and Hill's estimates $\widehat{\zeta}_{RS}$ and $\widehat{\zeta}_{Hill}$ are

$$(\widehat{G}_{RS} - 1.96 \cdot s.e._{\widehat{G}_{RS}}, \widehat{G}_{RS} + 1.96 \cdot s.e._{\widehat{G}_{RS}}) \tag{17}$$

and

$$(\widehat{G}_{Hill} - 1.96 \cdot s.e._{\widehat{G}_{Hill}}, \widehat{G}_{Hill} + 1.96 \cdot s.e._{\widehat{G}_{Hill}}). \tag{18}$$

Tables A.5 present the estimates $\widehat{G}_{RS}$ and $\widehat{G}_{Hill}$ of the Gini coefficient in the upper tails of the worth distribution in the World and the income distribution in Russia implied by the estimates $\widehat{\zeta}_{RS}$ and $\widehat{\zeta}_{Hill}$ Tables A.2.W and A.2.Ru. [11] In addition, the tables provide the standard errors of the estimates of the Gini coefficients and the corresponding confidence intervals derived as discussed above in the section.

The estimation results for the Gini coefficient in the upper tail of the wealth distribution in the World in Table A.5.W are largely in agreement with the benchmark *G*=0.5 implied the tail index value $\zeta = 1.5$ for the wealth distribution in the developed economies. In particular, the right-end points of the confidence intervals in Table A.5.W are close to 0.5. In addition, this value belongs to the confidence interval for 2009 constructed using Hill's tail index estimate. Thus, using the confidence interval, the hypothesis $H_0 : \zeta = 1.5$ is not rejected for the upper tail of the World's wealth distribution in that year. One should note that the agreement of estimates of

[11] We present the estimation results of the Gini coefficient for the World and Russia only since the estimates in Tables A.2 are the most reliable for these economies according to the sample sizes used in the estimation and the corresponding standard errors and confidence intervals. The analysis for the other economies considered can be conducted in a similar way.

the Gini coefficient in Table A.5.W with the value $\zeta = 1.5$ for the developed countries is somewhat less pronounced than in the comparisons of the tail index estimates in Table A.2.W with the benchmark value $\zeta = 1.5$. This, in part, is due to the fact that, as is easy to see, the standard errors $s.e._{\hat{G}} = \frac{2 \cdot s.e._{\hat{\zeta}}}{(2\hat{\zeta} - 1)^2}$ of $\hat{G}$ are smaller than the standard errors $s.e._{\hat{\zeta}}$ of $\hat{\zeta}$ if $\hat{\zeta} > 1.3$. Similarly, for such values of $\hat{\zeta}$, the length of the confidence intervals for G is smaller than the length of the corresponding confidence intervals for ζ.

Similar to the results for the tail index ζ in Table A.2.Ru, the semiparametric estimates of the Gini coefficient G for the upper tails (the upper 10%, 5% and 1%) of the income distribution in Russia in Table A.5.Ru are largely in agreement with the benchmark values $G \in (0.2, 0.5)$ for the developed economies. All of the point estimates $\widehat{G}$ are close to the value $G = 0.2$ that corresponds to $\zeta = 3$, and most of these estimates are slightly greater than 0.2 and thus belong to the interval $(0.2, 0.5)$. In addition, the confidence intervals for the Gini coefficient G in different upper tails (10%, 5% and 1%) in the table either lie in the interval (0.2, 0.5) or have their larger parts in it. The null hypothesis H_0: G =0.5 of large income inequality (corresponding to $\zeta = 1.5$) is rejected in favor of H_a: G <0.5 for all the time periods and tail truncation levels reported in the table.

It is interesting to compare the estimates of the Gini coefficient in the upper parts of the income distribution in Russia in Table A.5.Ru with the dynamics of the Gini coefficient for the whole distribution of income in this country (see Table B.2.Ru). Such comparisons are important because they provide information about inequality and the shape of the Lorenz curves in different parts of the underlying income distribution. Thus, importantly, the comparisons characterize the differences in the inequality among the relatively rich, the middle income and the relatively poor households.

According to Table B.2.Ru, the Gini coefficient for the whole income distribution in Russia increased from about 0.3 in 1992 to a rather stable value of about 0.4 in 1995-2009. These values belong to the benchmark interval $G \in (0.2, 0.5)$ for the upper tails of income distributions in the developed economies implied by the tail index estimates $\zeta \in (1.5, 3)$ in power law approximations (1) for income.

The value $G \approx 0.4$ for the Gini coefficient in the whole income distribution in Russia in 1995-2009 is, however, considerably greater than the estimates $G \approx 0.2$ of the Gini coefficient in the upper tails of the distribution in Table A.5.Ru. Moreover, according to the confidence intervals in Table A.5.Ru, the null hypothesis $H_0 : G = 0.4$ (and even the null hypothesis $H_0 : G = 0.3$) is rejected at the 2.5% significance level in favor of $H_0 : G < 0.4$ (resp., $H_0 : G < 0.3$) for the Gini coefficient in all the upper tails (10%, 5% and 1%) considered in the table. This implies that most of the income inequality in Russia is, apparently, due to the income disparities in the middle and the lower parts of the income distribution (this corresponds to a higher degree of convexity of the Lorenz curve for the middle and small income levels comparing to the large levels of income). In other words, apparently, the inequality in Russia is higher among the middle-income and relatively poor households than among the relatively rich households.

8. CONCLUSION AND FURTHER RESEARCH

Emerging and developing economies are likely to be more volatile than their developed counter-parts and subject to more extreme external and internal shocks. The higher degree of volatility leads to the expectation that heavy-tailedness properties and distributions of key variables in these markets, including income and wealth, may differ from those in developed economies. However, the results obtained in this paper point out to interesting and somewhat surprising similarities between the heavy-tailedness characteristics and distributional properties of income and wealth distributions in some of the post-Soviet economies, including Russia, and those in the developed markets. Among other important issues, these results characterize the equilibrium dynamics of income and wealth distributions in the markets considered and, as discussed in the paper, can be further used in the analysis of income and wealth inequality in them.

Among other directions, further research may focus on extensions of the analysis using larger samples of observations for some of the economies under the analysis. In particular, it is important to complement the empirical analysis for Central Asian economies in the paper using larger data samples. It is of interest to see whether the estimation results for the income and wealth distribution in the post-Soviet Central Asian countries, especially in Kyrgyzstan, Tajikistan and Uzbekistan, are similar to the results for the World obtained in the paper. The results on the tail index estimation in relatively small samples for the Central Asian countries considered in the paper may also be complemented using robust small sample tail index inference procedures such as those based on weighed Hill's estimators for different truncation levels in Huisman *et al.* (2001).

In addition, further research may take into account the contribution to income and wealth distributions from shadow wages and shadow incomes in the economies. Estimation of the shadow part of income may use, among others, the methods and approaches discussed, in the case of Uzbekistan, in Ibragimov, Ibragimov and Karimov (2010). The estimation results can be further applied to measurement of income inequality and poverty in Russia and Central Asian countries using both the official and shadow wages and incomes in the economies.

Further research developments may focus on the analysis of implications of the obtained empirical results for consumer demand on different classes of goods, including luxuries and necessities, and market equilibrium. This analysis can based on the theoretical results in Ibragimov and Ibragimov (1997) and their extensions to the case of heterogeneous preferences obtained in Appendix MD. In particular, it is of interest to consider implications of the empirical results for estimates of the volume of the unofficial part of emerging economies using the data on luxury goods consumption by households and estimates of elasticity of demand on luxuries.

The extensions may also develop further analysis of Weber-Fechner laws as alternatives to power law modeling for income distribution in the economies under consideration. Among other problems, it is of interest to explore the advantages of combining the Weber-Fechner and power law approaches for modeling both the middle of income distributions and its tails.

REFERENCES

Atkinson, A. B. (1970). On the measurement of inequality. *Journal of Economic Theory* **5**, 244-263.

Axtell, R. L. (2001). Zipf distribution of U.S. firm sizes. *Science* **293**, 1818-1820.

Bosker, M., Brakman, S., Garretsen, H., de Jong, H. and Schramm, M. (2006). The development of dities in Italy 1300 – 1861. *CESifo Working Paper* No. 1893, CESifo Group Munich.

Bosker, M., Brakman, S., Garretsen, H. and Schramm, M. (2008). A century of shocks: The evolution of the German city size distribution 1925-1999. *Regional Science and Urban Economics* **38**, 330-347.

Cowell, F. A. and Flachaire, E. (2007). Income distribution and inequality measurement: the problem of extreme values. *Journal of Econometrics* **141**, 1044-1072.

Dasgupta, P. A., Sen, A. and Starrett, D. (1973). Notes on the measurement of inequality. *Journal of Economic Theory* **6,** 180-187.

Davidson, R. and Flachaire, E. (2007). Asymptotic and bootstrap inference for inequality and poverty measures. *Journal of Econometrics* **141**, 141-166.

Drees, H., de Haan, L. and Resnick, S. (2000). How to make a Hill plot. *Annals of Statistics* **28**, 254-274.

Embrechts, P., Klüuppelberg, C. and Mikosch, T. (1997). *Modelling extremal events for insurance and finance*, Springer, New York.

Fama, E. (1965). The behavior of stock market prices. *Journal of Business* **38**, 34–105.

Gabaix, X. (1999). Zipf's law for cities: An explanation. *Quarterly Journal of Economics* 114, 739-767.

Gabaix, X. (2008). Power laws. *The New Palgrave Dictionary of Economics, 2nd Edition,* Eds. S. N. Durlauf and L. E. Blume. Palgrave Macmillan, New York.

Gabaix, X. (2009). Power laws in economics and finance. *Annual Review of Economics* **1**, 255-93.

Gabaix, X. and Ibragimov, R. (2011). Rank−1/2: A simple way to improve the OLS estimation of tail exponents. *Journal of Business and Economic Statistics* **29**, 24-39. Supplementary material: http://www.economics.harvard.edu/faculty/ibragimov/files/GabaixIbragimovAdditionalResults.pdf

Gabaix, X., Gopikrishnan, P., Plerou, V. and Stanley, H. E. (2003). A theory of power-law distributions in financial market fluctuations. *Nature* **423**, 267-270.

Gabaix, X., Gopikrishnan, P., Plerou, V. and Stanley, H. E. (2006). Institutional investors and stock market volatility. *Quarterly Journal of Economics* **121**, 461-504.

Gabaix, X. and Landier, A. (2008). Why has CEO pay increased so much? *Quarterly Journal of Economics* **123**, 49–100.

Granger, C.W.J. and Orr, D. (1972). Infinite variance and research strategy in time series analysis. *Journal of the American Statistical Association* **67**, 275–85.

Helpman, E., Melitz, M. J. and Yeaple, S. R. (2004). Export versus FDI with heterogeneous firms. *American Economic Review* **94**, 300–316.

Hinloopen, J. and van Marrewijk, C. (2006). Comparative advantage, the rank-size rule, and Zipf's law. *Tinbergen Institute Discussion Paper* 06-100/1.

Huisman, R., Koedijk, C.G., Palm, F. and Kool, C. (2001). Tail index estimates in small samples. *Journal of Business and Economic Statistics* 19, 208-216.

Ibragimov M. (1997). Disparity in income distribution in the countries of Central Asia. *Paper presented at the International Conference on Understanding the Transition in Central Asia, Macroeconomic Policy and Management Division (EDIMP), Economic Development Institute of the World Bank, June 1997, Washington, DC.*

Ibragimov, R. (2009). Heavy tailed densities. *The New Palgrave Dictionary of Economics Online*. Eds. S. N. Durlauf and L. E. Blume. Palgrave Macmillan, New York.

Ibragimov, M. and Ibragimov R. (2007). Market demand elasticity and income inequality. *Economic Theory* **32**, 579–587.

Ibragimov, M., Ibragimov, R. and Karimov, D. (2010). A method for estimation of shadow wages. *Working paper*, Harvard University. Available at http://ws1.ad.economics.harvard.edu/faculty/ibragimov/files/IbragimovIbragimovKarimov.pdf

Ibragimov, M. and Ibragimov, R. (2007). Market demand elasticity and income inequality. *Economic Theory* **32,** 579-587.

Ibragimov, M., Ibragimov, R. and Kattuman, P. (2009). Emerging markets and heavy tails. *Working paper*, Harvard University.

Ibragimov M., Ibragimov R. and Sirajiddinov N. (2008). Modeling and forecasting income tax revenue: the case of Uzbekistan. In *Income distribution: Inequalities, impacts and incentives* (Irwing H. Wadell, Ed.), Nova Science Publishers, Hauppage, NY.

Ibragimov, R., Jaffee, D. and Walden, J. (2009). Nondiversification traps in catastrophe insurance markets. *Review of Financial Studies* **22**, 959-993.

Ibragimov, R. and Mueller, U. K. (2010). *t*-statistic based correlation and heterogeneity robust inference. *Journal of Business and Economic Statistics* 28*,* 453-468. Also available as *Harvard Institute of Economic Research Discussion Paper* No. 2129, http://www.economics.harvard.edu/pub/hier/2007/HIER2129.pdf

Ioannides, Y. M., Overman, H. G., Rossi-Hansberg, E. and Schmidheiny, K. (2008). The effect of information and communication technologies on urban structure. *Economic Policy* **54**, 201-242.

Jansen, D. W. and de Vries, C. G. (1991). On the frequency of large stock returns: Putting booms and busts into perspective. *Review of Economics and Statistics* **73**, 18-32.

Lambert, P. J. and Pfähler, W. (1997). Market demand and income distribution: A theoretical exploration. *Bulletin of Economic Research* **49,** 137-151.

Levy, M. (2003). Are rich people smarter? *Journal of Economic Theory* **110**, 42–64.

Levy, M. and Levy, H. (2003). Investment talent and the Pareto wealth distribution: Theoretical and experimental analysis. *Review of Economics and Statistics* **85**, 709–725.

Lorenz, M. O. (1905). Methods of measuring concentration of wealth. *Journal of the American Statistical Association* **9,** 209-219.

Loretan, M. and Phillips, P. C. B. (1994). Testing the covariance stationarity of heavy-tailed time series. *Journal of Empirical Finance* **1**, 211-248.

Mandelbrot, B. (1963). The variation of certain speculative prices. *Journal of Business* **36**, 394–419.

Mandelbrot, B. (1997). *Fractals and Scaling in Finance*. Springer, New York.

Marshall, A. W., Olkin, I., (1979). Inequalities: Theory of Majorization and Its Applications. *New York: Academic Press*.

NSCKR, National Statistical Committee of Kyrgyz Republic (2009). Kyrgyzstan in numbers, Bishkek, Kyrgyzstan.

NSCKR, National Statistical Committee of Kyrgyz Republic (2010a). The statistical yearbook of Kyrgyz Republic: 2009, Bishkek, Kyrgyzstan.

NSCKR, National Statistical Committee of Kyrgyz Republic (2010b). The socioeconomic situation in Kyrgyz Republic: 2009, Bishkek, Kyrgyzstan.

Park, J. L., Hocomb, R. B., Raper, K. C. and Capps, O. (1996). A demand systems analysis of food commodities by U.S. households segmented by income. *American Journal of Agricultural Economics* **78**, 290-300.

Pinstrup-Andersen, P. and Cailedo, E., (1978). The potential impact of changes in income distribution on food demand and human nutrition. *American Journal of Agricultural Economics* **60**, 402-415.

Rosstat (2007). Rossia v tsifrakh (Russia in Numbers). *Statistical Bulletin,* the Federal State Statistics Service of Russian Federation, Moscow (in Russian).

Quadrini, V. (2008). Growth and inequality (macro perspectives). *The New Palgrave Dictionary of Economics, 2nd Edition,* Eds. S. N. Durlauf and L. E. Blume. Palgrave Macmillan.

Saposnik, R. (1993). A note on majorization theory and the evaluation of income distributions. *Economics Letters* **42,** 179-183.

Senauer, B. (1990). Household behavior and nutrition in developing countries. *Food Policy* **15**, 408-417

Shorrocks, A. F. (1983). Ranking income distributions. *Economica* **50,** 3-17.

Unnevehr, L. J. and Khoju, M. (1991). Economic development, income distribution and meat demand. *Journal of International Food and Agribusiness Marketing* **2**, 121 – 134.

Zhang, J. H., Chen, Q. H. and Wang, Y. G. (2009). Zipf distribution in top Chinese firms and an economic explanation. Physica A-Statistical Mechanics and Its Applications **388**, 2020-2024.

APPENDIX A: Estimation results

Table A.1.1: Descriptive statistics for distribution of income in households. Russia, 2005-2007 (quarterly data, average rubles per month)

Year, Quarter	Sample size	Mean	Median	Moda	Std. Dev.	Max.	Min.	Skew.	Kurt.	Gini coef.
2005,1	46974	22820	17075	10000	23940	1429393	0.02	14.5	570.1	0.414
2005,2	53132	25086	18975	10000	25290	1269834	0.25	10.3	294.4	0.412
2005,3	53129	27000	20302	9500	28032	1849601	0.00	13.2	539.3	0.415
2005,4	53135	29668	22430	13500	28740	1119837	0.04	8.6	210.6	0.412
2006,1	53093	28619	21793	9500	30548	2152383	0.07	20.6	1133.6	0.409
2006,2	53094	30445	23139	9500	33214	2110964	0.03	19.6	954.5	0.411
2006,3	53089	32435	24635	13500	31426	1981988	0.05	10.5	415.3	0.412
2006,4	53072	35784	27422	16000	32746	2104030	0.18	9.5	409.1	0.407
2007,1	50589	34561	26523	17000	34235	1718131	0.07	12.8	478.3	0.406
2007,2	49884	37562	28636	13000	34819	1869102	0.05	7.8	240.5	0.410
2007,3	53104	41355	31363	15000	47198	3246727	0.17	23.6	1285.2	0.418
2007,4	53096	46789	35585	18000	46460	2446446	0.06	10.0	305.6	0.413

Table A.1.2: Descriptive statistics for distribution of income in households, other CIS countries

Country	Year	Sample size	Mean	Median	Mode	Gini coefficient
Russia, rub	2007	53 096	15500	26200	46785	0.402
Kazakhstan, tenge	1997		2665	1230		0.497
	2009		34736			0.267
Kyrgyzstan, som	1994	4 489 300	138	110	70	0.32
Uzbekistan, sum	1997	36 039	2054	1140	900	0.457
Tajikistan, somoni	2009	20	304.78	181	148	0.427

Note: The data for Kazakhstan and Tajikistan in 2009 are the nominal average salaries over economic industries. Similarly, the data for Kazakhstan in 1997 is based on income intervals.

Table A.2.W: Tail index estimates for the World.

1	2	3	4	5	6	7	8
Year	**Sample Size**	$\hat{\zeta}_{RS}$	$s.e._{RS} = \sqrt{\frac{2}{n}}\hat{\zeta}_{RS}$	95% CI, equation (12)	$\hat{\zeta}_{Hill}$	$s.e._{Hill} = \frac{1}{\sqrt{n}}\hat{\zeta}_{Hill}$	95% CI, equation (4)
2008	102	2.109	0.295	(1.530, 2.688)	1.8601	0.1842	(1.499, 2.221)
2009	103	1.944	0.271	(1.414, 2.476)	1.6868	0.1662	(1.361, 2.013)

Diagram A.2.W: Log-log rank-size plots for the World, 2008-2009

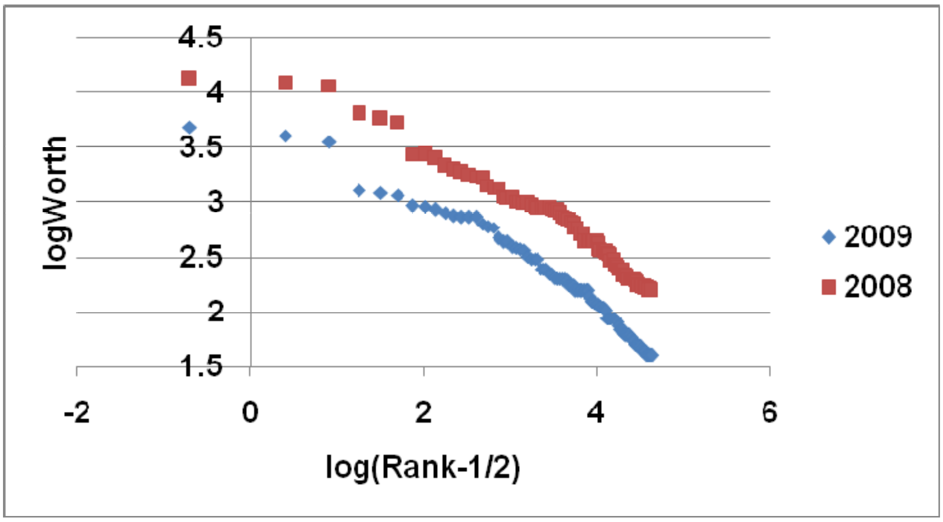


Table A.2.Ru: Tail index estimates for Russia.

1	2	3	4	5	6	7	8	9	10
Year, quarter	**Sample size, *N***	**% of largest observations, *m***	***n***	$\hat{\zeta}_{RS}$	$s.e._{RS} = \sqrt{\frac{2}{n}}\hat{\zeta}_{RS}$	**95% CI, equation (12)**	$\hat{\zeta}_{Hill}$	$s.e._{Hill} = \frac{1}{\sqrt{n}}\hat{\zeta}_{Hill}$	**95% CI, equation (4)**
2005,1	46974	10	4697	2.942	0.061	(2.823, 3.060)	2.827	0.041	(2.746, 2.908)
		5	2349	2.950	0.086	(2.781, 3.118)	3.012	0.062	(2.890, 3.134)
		1	470	2.457	0.160	(2.143, 2.771)	3.136	0.145	(2.852, 3.420)
2005,2	53132	10	5313	2.887	0.056	(2.777, 2.996)	2.786	0.038	(2.711, 2.861)
		5	2656	2.916	0.080	(2.760, 3.073)	2.997	0.058	(2.883, 3.111)
		1	531	2.514	0.154	(2.211, 2.816)	3.062	0.133	(2.802, 3.323)
2005,3	53129	10	5313	2.871	0.056	(2.762, 2.980)	2.755	0.038	(2.681, 2.829)
		5	2656	2.877	0.079	(2.722, 3.031)	2.972	0.058	(2.859, 3.085)
		1	531	2.558	0.157	(2.250, 2.865)	2.758	0.120	(2.524, 2.993)
2005,4	53135	10	5313	3.045	0.059	(2.929, 3.161)	2.879	0.040	(2.801, 2.956)
		5	2656	3.075	0.084	(2.909, 3.240)	3.202	0.062	(3.080, 3.324)
		1	531	2.625	0.161	(2.310, 2.941)	3.174	0.138	(2.904, 3.444)
2006,1	53093	10	5309	2.999	0.058	(2.885, 3.113)	2.873	0.039	(2.796, 2.950)
		5	2655	2.971	0.082	(2.811, 3.131)	3.199	0.062	(3.077, 3.321)
		1	531	2.388	0.147	(2.101, 2.675)	3.145	0.137	(2.878, 3.413)
2006,2	53094	10	5309	2.880	0.056	(2.771, 2.990)	2.853	0.039	(2.777, 2.930)
		5	2655	2.828	0.078	(2.676, 2.980)	2.992	0.058	(2.878, 3.106)
		1	531	2.395	0.147	(2.107, 2.683)	2.887	0.125	(2.642, 3.133)
2006,3	53089	10	5309	2.994	0.058	(2.880, 3.108)	2.792	0.038	(2.717, 2.867)
		5	2655	3.076	0.084	(2.910, 3.241)	3.048	0.059	(2.932, 3.164)
		1	531	2.947	0.181	(2.593, 3.302)	3.224	0.140	(2.949, 3.498)
2006,4	53072	10	5309	3.238	0.063	(3.115, 3.361)	2.886	0.040	(2.808, 2.963)
		5	2655	3.489	0.096	(3.301, 3.677)	3.237	0.063	(3.114, 3.360)
		1	531	3.467	0.213	(3.050, 3.883)	3.904	0.169	(3.572, 4.236)
2007,1	50589	10	5059	3.019	0.060	(2.902, 3.137)	2.891	0.041	(2.811, 2.970)
		5	2530	3.023	0.085	(2.857, 3.190)	3.109	0.062	(2.988, 3.231)
		1	506	2.599	0.163	(2.279, 2.919)	2.855	0.127	(2.606, 3.104)
2007,2	49884	10	4988	3.088	0.062	(2.967, 3.210)	2.857	0.041	(2.778, 2.937)
		5	2500	3.257	0.092	(3.076, 3.437)	3.094	0.062	(2.973, 3.215)
		1	499	3.322	0.210	(2.910, 3.734)	3.528	0.158	(3.219, 3.838)
2007,3	53104	10	5310	2.912	0.057	(2.801, 3.023)	2.835	0.039	(2.759, 2.911)
		5	2655	2.904	0.080	(2.748, 3.060)	3.009	0.058	(2.894, 3.123)
		1	531	2.412	0.148	(2.122, 2.702)	3.240	0.141	(2.965, 3.516)
2007,4	53096	10	5310	2.941	0.057	(2.829, 3.053)	2.811	0.039	(2.735, 2.886)
		5	2655	2.972	0.082	(2.812, 3.131)	3.077	0.060	(2.960, 3.194)
		1	531	2.626	0.161	(2.310, 2.941)	2.935	0.127	(2.685, 3.184)

Diagram A.2.Ru: Log-log rank-size plot for Russia in 2005-2007 (1% tail truncation level).

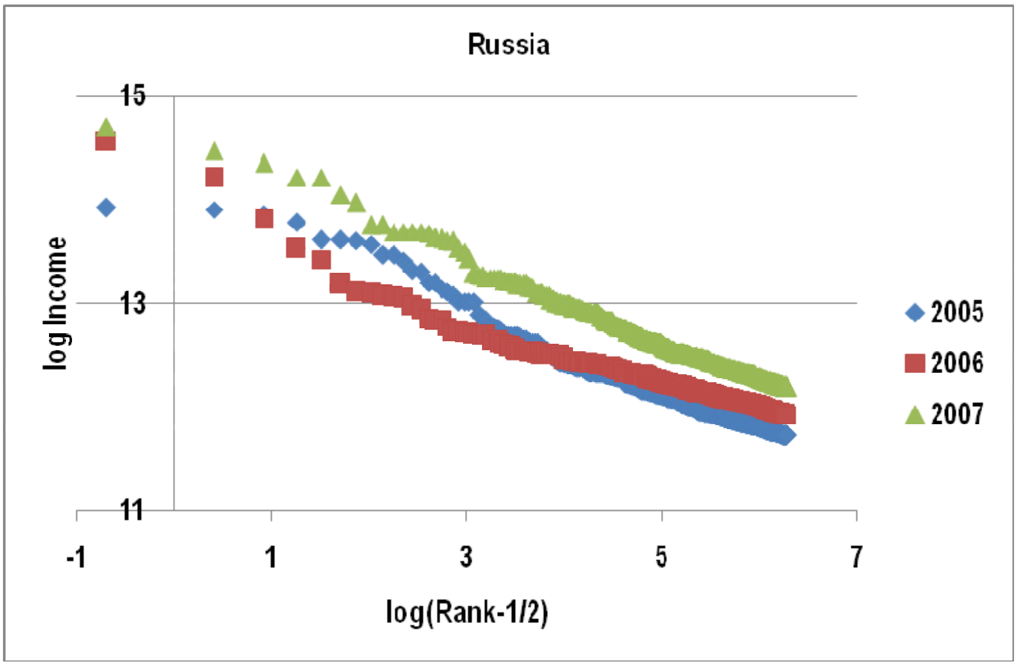


Table A.2.Kz: Tail index estimates for Kazakhstan.

Years	Sample size, N	% of largest observations, m	n	$\hat{\zeta}_{RS}$	$s.e._{RS} = \sqrt{\frac{2}{n}}\hat{\zeta}_{RS}$	95% CI, equation (12)	$\hat{\zeta}_{Hill}$	$s.e._{Hill} = \frac{1}{\sqrt{n}}\hat{\zeta}_{Hill}$	95% CI, equation (4)
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2006	46	50	23	2.792	0.823	(1.178, 4.406)	3.273	0.682	(1.935, 4.610)
		20	9	2.301	1.085	(0.175, 4.426)	3.096	1.032	(1.073, 5.119)
2007	46	50	23	3.051	0.900	(1.288, 4.815)	3.679	0.767	(2.175, 5.182)
		20	9	2.478	1.168	(0.188, 4.767)	3.025	1.008	(1.049, 5.001)
2008	46	50	23	3.037	0.896	(1.282, 4.793)	3.743	0.781	(2.213, 5.273)
		20	9	2.575	1.214	(0.196, 4.955)	3.147	1.049	(1.091, 5.203)

Note: Estimates are based on the nominal average salaries in economic industries.

Diagram A.2.Kz: Log-log rank-size plots for Kazakhstan.

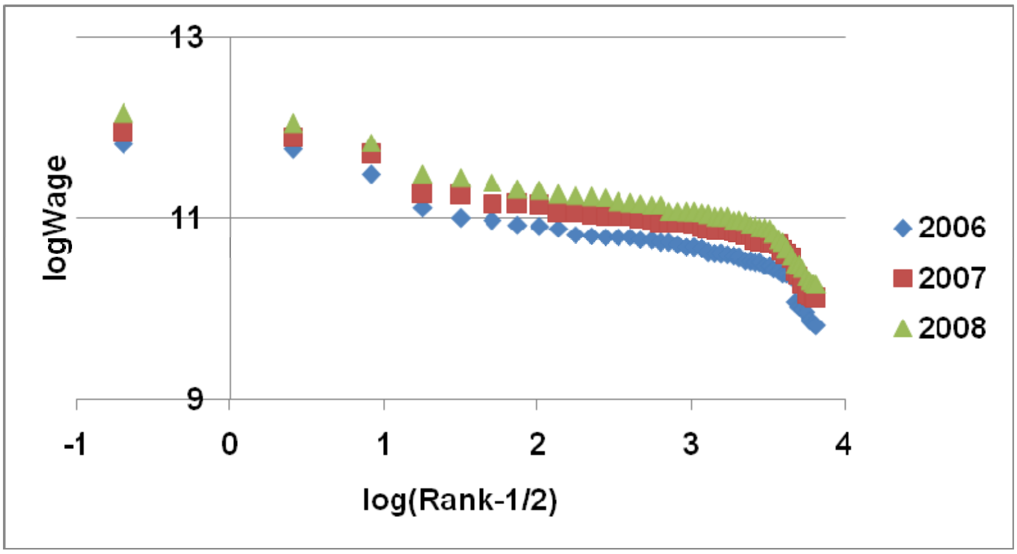


Table A.2.Kg: Tail index estimates for Kyrgyzstan

Years	% of largest observations, *m*	*n*	$\hat{\zeta}_{RS}$	$s.e._{RS} = \sqrt{\frac{2}{n}}\hat{\zeta}_{RS}$	95% CI, equation (12)	$\hat{\zeta}_{Hill}$	$s.e._{Hill} = \frac{1}{\sqrt{n}}\hat{\zeta}_{Hill}$	95% CI, equation (4)
1	2	3	4	5	6	7	8	9
1999	100	15	1.536	0.561	(0.437, 2.636)	1.1269	0.2910	(0.557, 1.697)
	50	8	2.988	1.494	(0.060, 5.917)	3.5602	1.2587	(1.093, 6.027)
2000	100	15	1.300	0.475	(0.370, 2.230)	1.1156	0.2881	(0.551, 1.680)
	50	8	1.675	0.838	(0.034, 3.317)	2.6129	0.9238	(0.802, 4.424)
2001	100	15	1.276	0.466	(0.363, 2.189)	1.0157	0.2622	(0.502, 1.530)
	50	8	1.673	0.837	(0.033, 3.313)	2.3103	0.8168	(0.709, 3.911)
2002	100	15	1.499	0.547	(0.426, 2.572)	1.2180	0.3145	(0.602, 1.834)
	50	8	2.905	1.453	(0.058, 5.753)	3.1615	1.1178	(0.971, 5.352)
2003	100	15	1.513	0.552	(0.430, 2.595)	1.2229	0.3157	(0.604, 1.842)
	50	8	2.946	1.473	(0.059, 5.834)	3.0088	1.0638	(0.924, 5.094)
2004	100	15	1.512	0.552	(0.430, 2.594)	1.0795	0.2787	(0.533, 1.626)
	50	8	2.479	1.240	(0.050, 4.909)	2.6975	0.9537	(0.828, 4.567)
2005	100	15	1.488	0.543	(0.423, 2.553)	1.0775	0.2782	(0.532, 1.623)
	50	8	2.314	1.157	(0.046, 4.581)	3.0103	1.0643	(0.924, 5.096)
2006	100	15	1.586	0.579	(0.451, 2.721)	1.0752	0.2776	(0.531, 1.619)
	50	8	2.289	1.144	(0.046, 4.532)	2.4848	0.8785	(0.763, 4.207)
2007	100	15	1.619	0.591	(0.460, 2.777)	1.0446	0.2697	(0.516, 1.573)
	50	8	2.791	1.395	(0.056, 5.525)	3.2765	1.1584	(1.006, 5.547)
2008	100	15	1.518	0.554	(0.432, 2.605)	0.9497	0.2452	(0.469, 1.430)
	50	8	2.532	1.266	(0.051, 5.013)	3.1190	1.1027	(0.958, 5.280)

Note: Estimates are based on the nominal average salaries in branches of the economy.

Diagram A.2.Kg: Log-log rank-size plots for Kyrgyzstan.

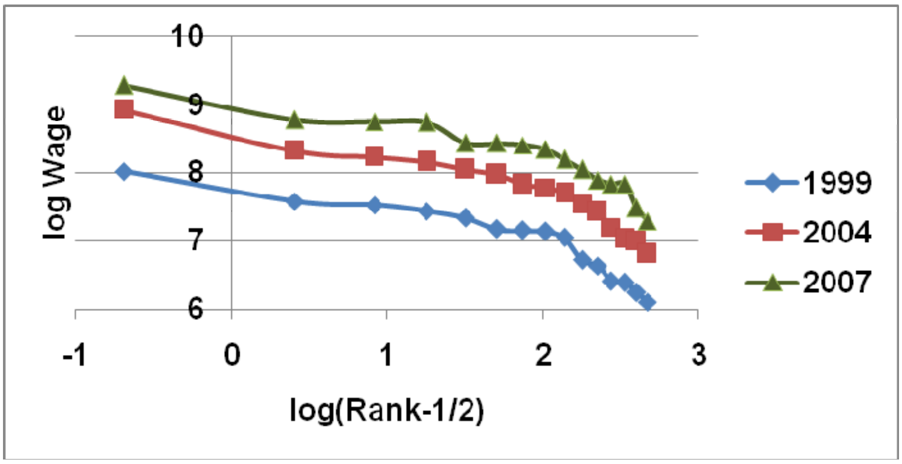


Table A.2.Tj: Tail index estimates for Tadjikistan

Years	Sample size	% of largest observations	n	$\hat{\zeta}_{RS}$	$s.e._{RS} = \sqrt{\frac{2}{n}}\hat{\zeta}_{RS}$	95% CI, equation (12)	$\hat{\zeta}_{Hill}$	$s.e._{Hill} = \frac{1}{\sqrt{n}}\hat{\zeta}_{Hill}$	95% CI, equation (4)
1	2	3	4	5	6	7	8	9	10
2000	20	100	20	1.165	0.369	(0.443, 1.887)	0.859	0.192	(0.483, 1.236)
		50	10	1.698	0.759	(0.210, 3.186)	2.141	0.677	(0.814, 3.468)
2009	20	100	20	1.188	0.376	(0.452, 1.925)	0.729	0.163	(0.410, 1.049)
		50	10	2.157	0.965	(0.266, 4.047)	2.199	0.695	(0.836, 3.561)

Note: Estimates are based on the nominal average salaries in branches of the economy.

Diagram A.2.Tj: Log-log rank-size plots for Tajikistan.

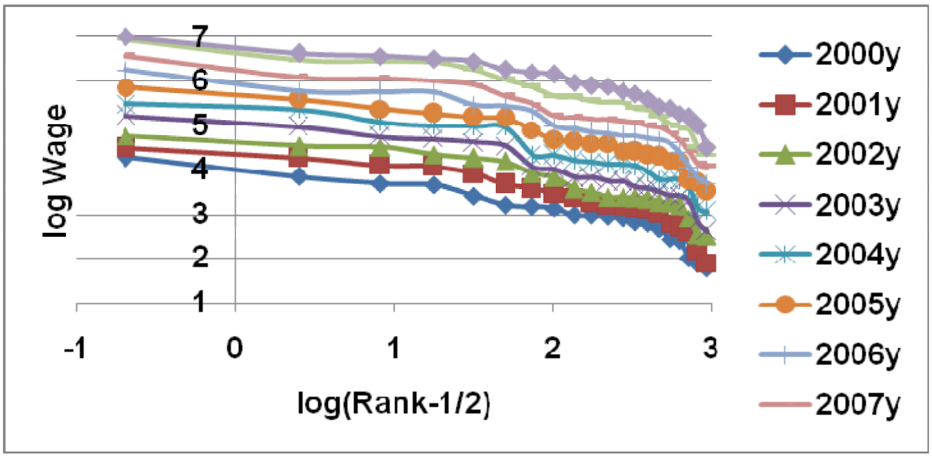


Table A.2.Uz: Tail index estimates for Uzbekistan

Years	Sample size	% of largest observations	n	$\hat{\zeta}_{RS}$	$s.e._{RS} = \sqrt{\frac{2}{n}}\hat{\zeta}_{RS}$	95% CI, equation (12)	$\hat{\zeta}_{Hill}$	$s.e._{Hill} = \frac{1}{\sqrt{n}}\hat{\zeta}_{Hill}$	95% CI, equation (4)
1	2	3	4	5	6	7	8	9	10
2000	20	100	20	1.640	0.517	(0.623, 2.656)	1.092	0.244	(0.613, 1.570)
		50	10	3.811	1.704	(0.470, 7.150)	3.175	1.004	(1.207, 5.142)
2001	20	100	20	1.758	0.556	(0.668, 2.847)	1.140	0.255	(0.640, 1.639)
		50	10	4.652	2.080	(0.574, 8.729)	5.238	1.656	(1.991, 8.485)
2002	20	100	20	1.702	0.538	(0.647, 2.757)	1.205	0.269	(0.677, 1.733)
		50	10	4.383	1.960	(0.541, 8.225)	4.350	1.376	(1.654, 7.046)
2003	20	100	20	1.611	0.509	(0.612, 2.610)	1.202	0.269	(0.675, 1.728)
		50	10	3.121	1.396	(0.385, 5.856)	3.080	0.974	(1.171, 4.989)
2004	20	100	20	1.338	0.423	(0.509, 2.167)	0.758	0.169	(0.426, 1.090)
		50	10	3.408	1.524	(0.421, 6.396)	3.479	1.100	(1.323, 5.635)
2005	20	100	20	1.356	0.429	(0.516, 2.197)	0.706	0.158	(0.397, 1.016)
		50	10	3.119	1.395	(0.385, 5.852)	3.124	0.988	(1.188, 5.061)
2006	20	100	20	1.418	0.448	(0.539, 2.297)	0.725	0.162	(0.407, 1.042)
		50	10	2.926	1.309	(0.361, 5.491)	2.978	0.942	(1.132, 4.824)

Note: Estimates are based on the nominal average salaries in branches of the economy.

Table A.3. Weber-Fechner law: linear-log size-rank regressions (8)-(9).

	World		**Russia**	**Kazakhstan**
	2008	**2009**		
a	3.4834	2.9984	11.7659	6.0980
	(0.0309)	(0.0353)	(0.0021)	(0.0553)
b	0.0145	0.0151	4.94E-05	0.0888
	(0.0005)	(0.0006)	(6.92E-08)	(0.0025)
R^2	0.886	0.867	0.906	0.972
$F(R^2)$	779.35	659.91	510381.4	1227.06
n	102	103	53096	37

Note: Standard errors of the regression coefficients are given in brackets. The *p*-values of the coefficients are smaller than 0.00005. The *p*-value for R^2 is smaller than 0.0000005; $F((R^2)$ denotes the value of the corresponding Fisher statistics.

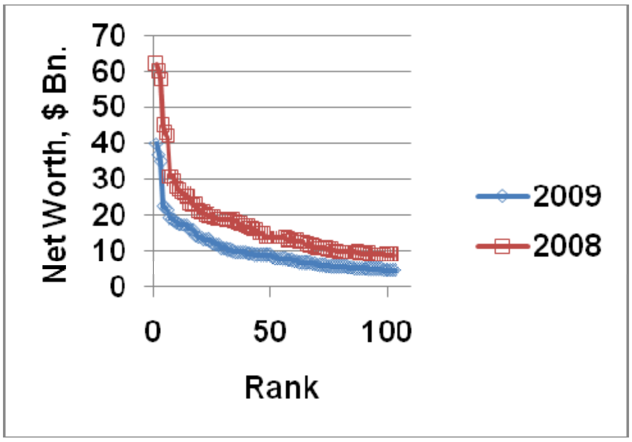


Diagram A.3.W: Weber-Fechner law for the World

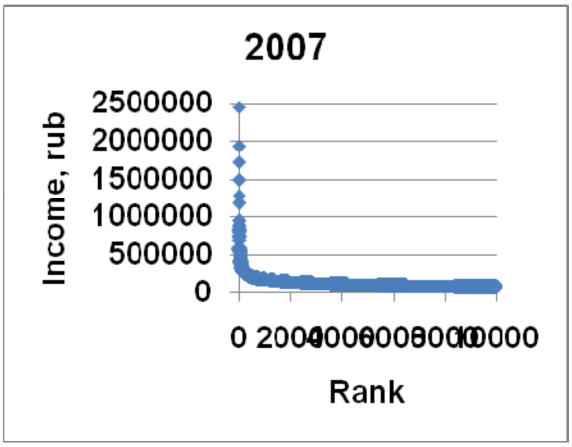
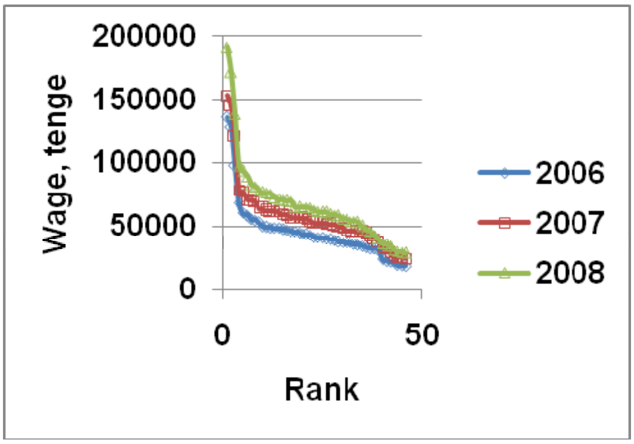


Diagram A.3.Ru: Weber-Fechner law for Russia



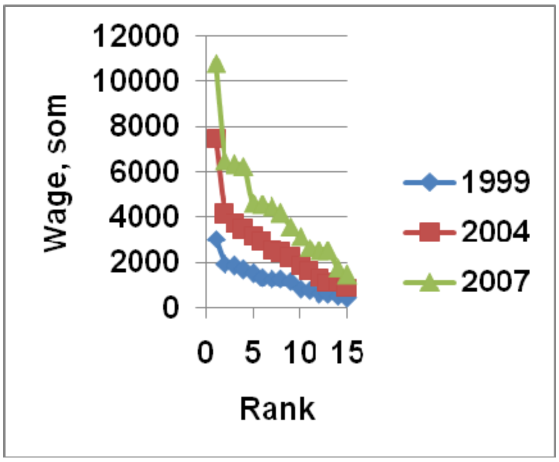


Diagram A.3.Kz: Weber-Fechner law for Kazakhstan

Diagram A.3.Kg: Weber-Fechner law for Kyrgyzstan

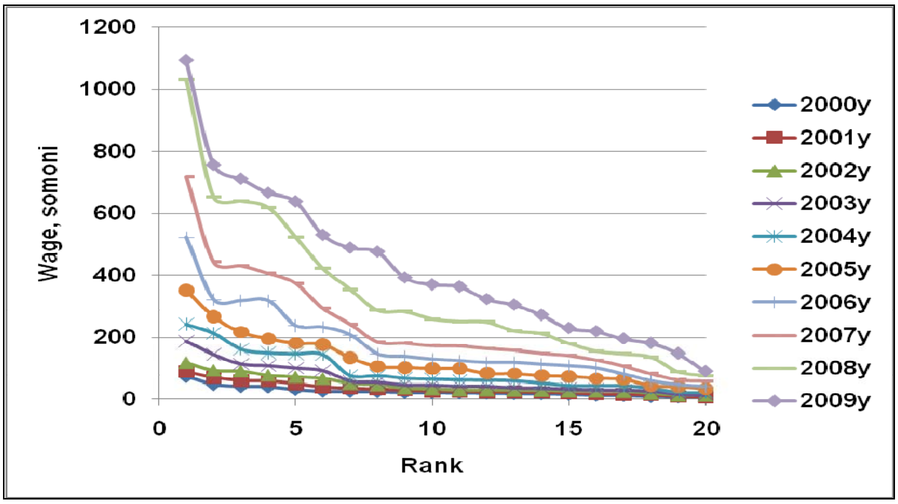


Diagram A.3.Tj: Weber-Fechner law for Tajikistan

Table A.4: Estimates for hierarchy of logarithms regression (8)-(9) with m=3:

	World	
	2008	**2009**
a	0.2459	0.1628
	(0.0054)	(0.0128)
b	−0.0051	−0.0088
	(9.18E-05)	(0.0002)
R^2	0.969	0.944
$F(R^2)$	3135.37	1699.59
n	102	103

Note: Standard errors of the regression coefficients are given in brackets. The coefficients are significant if the significance level is above 0.00005. The p-value for R^2 is smaller than 0.0000005; $F(R^2)$ denotes the value of the corresponding Fisher statistics. The form of the hierarchy of logarithms with the number of iterations m=3 provides the best fit among other choices for m considered in the analysis.

The following are approximations to the distribution of the worth in the data implied by the estimates in Table A.4:

$$2008\text{: } Worth = \exp(\exp(\exp(0.245859 - 0.005138 \cdot Rank)))$$

$$2009\text{: } Worth = \exp(\exp(\exp(0.162768 - 0.008812 \cdot Rank)))\ .$$ [12]

Table A.5.W: Semiparametric estimates of the Gini coefficient in the upper tails of the worth distribution in the World.

Year	**Sample size**	$\widehat{G}_{RS}$	$s.e._{\widehat{G}_{RS}}$	**95% CI, equation (17)**	$\widehat{G}_{Hill}$	$s.e._{\widehat{G}_{Hill}}$	**95% CI, equation (18)**
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
2008	102	0.311	0.057	(0.199, 0.422)	0.368	0.050	(0.270, 0.465)
2009	103	0.346	0.065	(0.219, 0.474)	0.421	0.059	(0.306, 0.537)

[12] Interpretation of such approximations obtained from the hierarchy of logarithms and their conclusions for income distributions is not yet clear and is left for further research.

Table A.5.Ru: Semiparametric estimates of the Gini coefficient in the upper tails of income distribution in Russia.

Year, quarter	Sample size, N	% of largest observations	n	$\widehat{G}_{RS}$	$s.e._{\widehat{G}_{RS}}$	95% CI, equation (17)	$\widehat{G}_{Hill}$	$s.e._{\widehat{G}_{Hill}}$	95% CI, equation (18)
1	2	3	4	5	6	7	8	9	10
2005,1	46974	10	4697	0.205	0.005	(0.195, 0.215)	0.215	0.004	(0.207, 0.222)
		5	2349	0.204	0.007	(0.190, 0.218)	0.199	0.005	(0.189, 0.209)
		1	470	0.256	0.021	(0.214, 0.297)	0.190	0.010	(0.169, 0.210)
2005,2	53132	10	5313	0.210	0.005	(0.200, 0.219)	0.219	0.004	(0.212, 0.226)
		5	2656	0.207	0.007	(0.193, 0.220)	0.200	0.005	(0.191, 0.209)
		1	531	0.248	0.019	(0.211, 0.286)	0.195	0.010	(0.175, 0.215)
2005,3	53129	10	5313	0.211	0.005	(0.201, 0.221)	0.222	0.004	(0.214, 0.229)
		5	2656	0.210	0.007	(0.197, 0.224)	0.202	0.005	(0.193, 0.212)
		1	531	0.243	0.019	(0.207, 0.279)	0.221	0.012	(0.198, 0.244)
2005,4	53135	10	5313	0.196	0.005	(0.188, 0.205)	0.210	0.003	(0.203, 0.217)
		5	2656	0.194	0.006	(0.182, 0.207)	0.185	0.004	(0.177, 0.193)
		1	531	0.235	0.018	(0.200, 0.270)	0.187	0.010	(0.168, 0.206)
2006,1	53093	10	5309	0.200	0.005	(0.191, 0.209)	0.211	0.003	(0.204, 0.218)
		5	2655	0.202	0.007	(0.189, 0.215)	0.185	0.004	(0.177, 0.194)
		1	531	0.265	0.021	(0.225, 0.305)	0.189	0.010	(0.170, 0.208)
2006,2	53094	10	5309	0.210	0.005	(0.200, 0.220)	0.212	0.004	(0.206, 0.219)
		5	2655	0.215	0.007	(0.201, 0.229)	0.201	0.005	(0.191, 0.210)
		1	531	0.264	0.020	(0.224, 0.304)	0.209	0.011	(0.188, 0.231)
2006,3	53089	10	5309	0.200	0.005	(0.191, 0.210)	0.218	0.004	(0.211, 0.225)
		5	2655	0.194	0.006	(0.182, 0.207)	0.196	0.005	(0.187, 0.205)
		1	531	0.204	0.015	(0.175, 0.234)	0.184	0.009	(0.165, 0.202)
2006,4	53072	10	5309	0.183	0.004	(0.174, 0.191)	0.210	0.003	(0.203, 0.216)
		5	2655	0.167	0.005	(0.157, 0.178)	0.183	0.004	(0.174, 0.191)
		1	531	0.169	0.012	(0.145, 0.192)	0.147	0.007	(0.133, 0.161)
2007,1	50589	10	5059	0.198	0.005	(0.189, 0.208)	0.209	0.004	(0.202, 0.216)
		5	2530	0.198	0.007	(0.185, 0.211)	0.192	0.005	(0.183, 0.201)
		1	506	0.238	0.019	(0.202, 0.275)	0.212	0.011	(0.190, 0.235)
2007,2	49884	10	4988	0.193	0.005	(0.184, 0.202)	0.212	0.004	(0.205, 0.219)
		5	2500	0.181	0.006	(0.170, 0.193)	0.193	0.005	(0.184, 0.202)
		1	499	0.177	0.013	(0.151, 0.203)	0.165	0.009	(0.148, 0.182)
2007,3	53104	10	5310	0.207	0.005	(0.198, 0.217)	0.214	0.004	(0.207, 0.221)
		5	2655	0.208	0.007	(0.194, 0.222)	0.199	0.005	(0.190, 0.208)
		1	531	0.262	0.020	(0.222, 0.301)	0.182	0.009	(0.164, 0.201)
2007,4	53096	10	5310	0.205	0.005	(0.195, 0.214)	0.216	0.004	(0.209, 0.223)
		5	2655	0.202	0.007	(0.189, 0.215)	0.194	0.004	(0.185, 0.203)
		1	531	0.235	0.018	(0.200, 0.270)	0.205	0.011	(0.184, 0.226)

APPENDIX B: Frequency distributions and cdf's

B.1. Frequency distributions

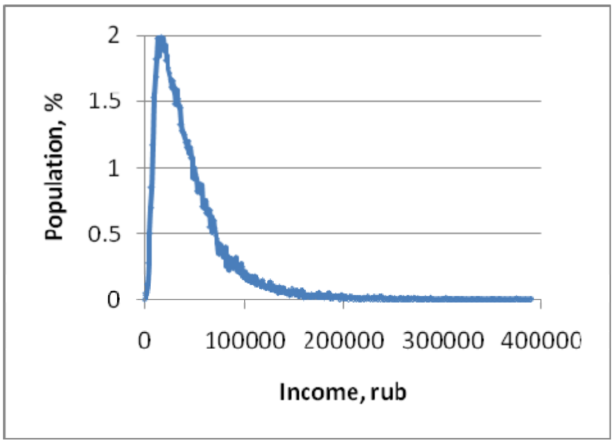


Diagram B.1.Ru: Russia

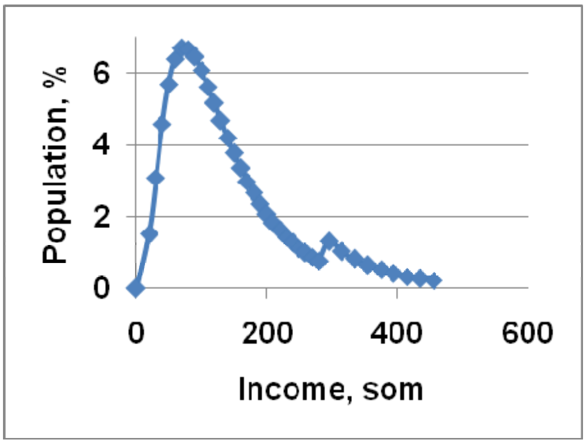


Diagram B.1.Kg: Kyrgyzstan

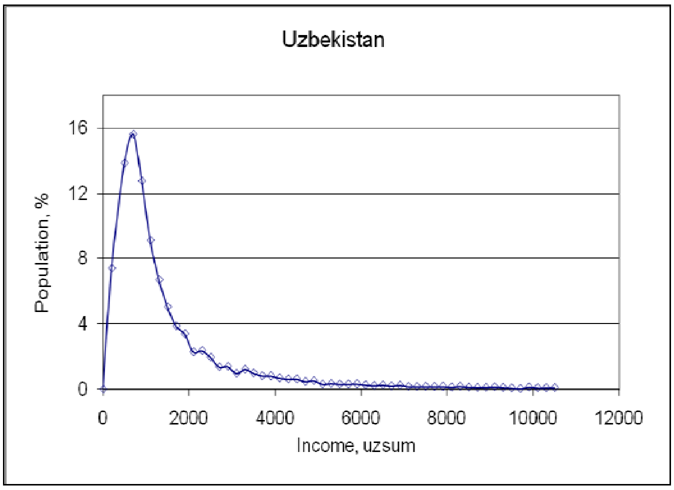


Diagram B.1.Uz: Uzbekistan

B.2. Cdf's

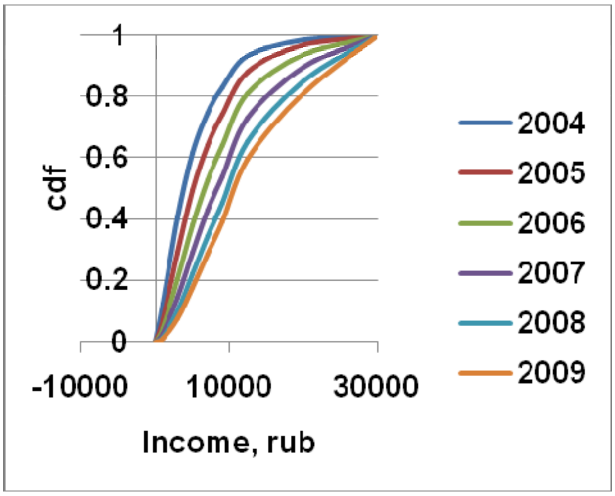


Diagram B.2.Ru: Russia

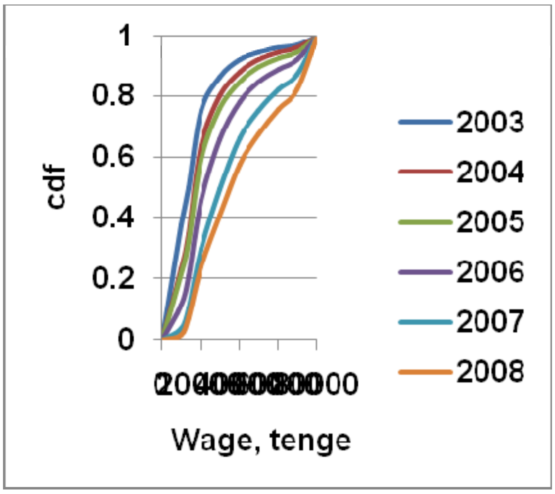


Diagram B.2.Kz: Kazakhstan

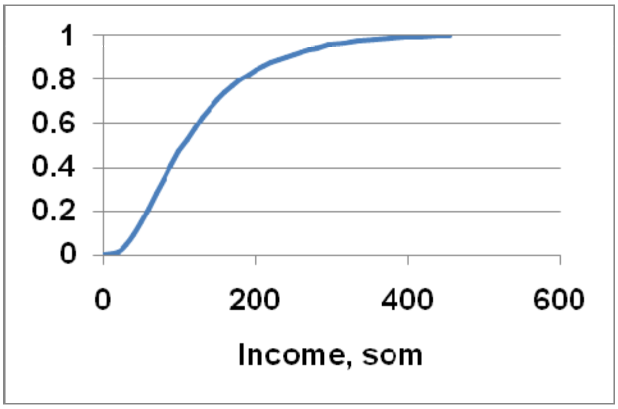


Diagram B.2.Kg: Kyrgyzstan

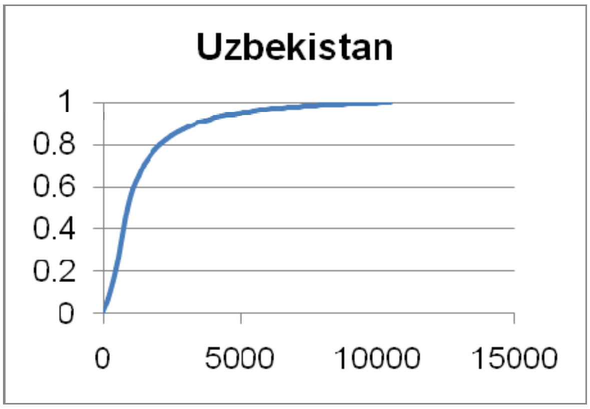


Diagram B.2.Uz: Uzbekistan

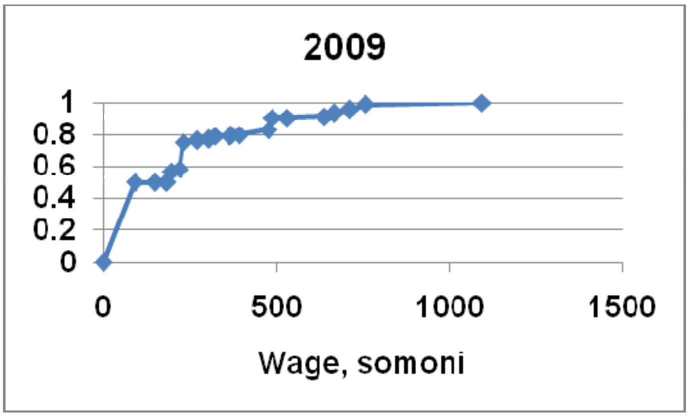


Diagram B.1.TJ: Tajikistan

Table B.2.Ru: The Gini coefficient in Russia.

Years	1992	1995	2000	2004	2005	2006	2007	2008	2009
Gini coefficient	0,289	0,387	0,395	0,409	0,409	0,416	0,423	0,422	0,422

Source: Rosstat, the Federal State Statistics Service of Russian Federation.

Table B.2.Kz: The Gini coefficient in Kazakhstan

Years	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
Gini coefficient	0.319	0.338	0.347	0.332	0.307	0.339	0.328	0.315	0.305	0.304	0.312	0.309	0.288	0.267

Source: The Statistics Agency of Kazakhstan

Table B.2.Uz: The Gini coefficient in Uzbekistan

Years	2000	2001	2002	2003	2004	2005	2006
Gini coefficient	0,423	0,415	0,423	0,441	0,416	0,402	0,382

Note: The estimates by M. Ibragimov using the data from the State Committee on Statistics of the Republic of Uzbekistan

Table B.2.Kg: The Gini coefficient in Kyrgyzstan

Years	2005	2006	2007
Gini coefficient	0,433	0,446	0,442

Source: The State Statistics Committee of Kyrgyz Republic

APPENDIX MD: Income distribution and market demand: The case of heterogeneous preferences

In recent years, a number of studies have focused on modeling income inequality using majorization relation (see, e.g., Marshall and Olkin, 1979) and applications of the latter concept to the problems in economics. The approach to the analysis of income inequality based on majorization which dates back to Lorenz (1905) has been used, among others, by Atkinson (1970), Dasgupta, Sen and Starrett (1973), Shorrocks (1983) and, more recently, Saposnik (1993). Using related concepts and methods, Lambert and Pfahler (1997) presented an analysis of the effects of income (re-)distribution on the market demand for a good or service.

In Ibragimov and Ibragimov (2007), the authors applied majorization theory to study dependence of market demand elasticity on the inequality in income distribution among the consumers. However, in that work it is assumed that consumers' preferences are the same for given prices on goods independently of their income levels. In this note, we extend the results obtained in Ibragimov and Ibragimov (2007) to the case where consumers' preferences are heterogeneous and the condition on equality of individual demand functions does not necessarily hold. This case is more realistic because consumers' preferences are affected by a variety of different factors.

Let there be K consumers and M goods in an economy. Denote by $\phi_{mk}(P,I_k)$ the function of the kth consumer's demand on the mth good, by $I=(I_1,\dots,I_K)$ the vector of incomes of the consumers and by $P=(p_1,\dots,p_M)$ the vector of prices on goods.

Let $\Phi_m(P,I)=\sum_{k=1}^{K}\phi_{mk}(P,I_k)$ be the function of market (aggregate) demand on good m and let $e_m(I)=\partial \log\Phi_m(P,I)/\partial \log p_m$ stand for its own-price elasticity. Denote by $S_{mk}\subset \mathbf{R}^{m+1}$ the domain of definition of the function $\phi_{mk}(P,I_k)$ and by $S_m=\{(P,I)=(P,I_1,\dots,I_k)\in\mathbf{R}^{M+K},\ (P,I_k)\in S_{mk},\ k=1,\dots,K\}$ the domain of definition of the function $\Phi_m(P,I)$, $m=1,\dots,M$.

According to the idea going back to Lorenz (1905) (see Marshall and Olkin, 1979), a vector $I^{(1)}=(I_1^{(1)},I_2^{(1)},\dots,I_K^{(1)})$ represents a more uniform distribution of the total income Y among K consumers than a vector $I^{(2)}=(I_1^{(2)},I_2^{(2)},\dots,I_K^{(2)})$ if $\sum_{i=1}^{l}I_{[i]}^{(1)}\le\sum_{i=1}^{l}I_{[i]}^{(2)}$, $l=1,\dots,K-1$, and $\sum_{i=1}^{K}I_{[i]}^{(1)}=\sum_{i=1}^{K}I_{[i]}^{(2)}=Y$, where $I_i^{(j)}$, $j=1,2$, are the income levels of the ith consumer and

$I_{[1]}^{(j)} \geq I_{[2]}^{(j)} \geq ... \geq I_{[K]}^{(j)}$ denote the components of the vectors $I^{(j)}$, j=1, 2, in decreasing order (if the above conditions hold, it is said that the vector $I^{(1)} = (I_1^{(1)}, I_2^{(1)}, ..., I_K^{(1)})$ is majorized by $I^{(2)} = (I_1^{(2)}, I_2^{(2)}, ..., I_K^{(2)})$, written $I^{(1)} \prec I^{(2)}$).

A function *f*(*I*) is called *Schur-convex* (resp., *Schur-concave)* in *I* if $I^{(1)} \prec I^{(2)} \Rightarrow (f(I^{(1)}) \leq f(I^{(2)}))$ (resp. $I^{(1)} \prec I^{(2)} \Rightarrow (f(I^{(1)}) \geq f(I^{(2)}))$).

Theorem MD.1 *(i) Let the individual demand functions* ($\phi_{mk}(P,I_k)$) *be twice continuously differentiable and let, for all* $(P,I) \in S_m$ *such that* $I_r \leq I_s$, *the following conditions hold:*

$$\frac{\partial \phi_{mr}(P,I_r)}{\partial I_r} \leq \frac{\partial \phi_{mr}(P,I_s)}{\partial I_s} \qquad \text{(MD.1)}$$

$$\frac{\partial^2 \phi_{mr}(P,I_r)}{\partial p_m \partial I_r} \leq \frac{\partial^2 \phi_{mr}(P,I_s)}{\partial p_m \partial I_s} \qquad \text{(MD.2)}$$

where p_m *is the price of the mth good in consideration. Then the absolute value of the elasticity* $|e_m(I)|$ *is Schur-concave in I on the set* S_m. *That is, the more non-uniform is the distribution of the total income among consumers in the economy, the smaller is the elasticity of the aggregate demand on the considered good by the absolute value.*

(ii) If in conditions (MD.1) and (MD.2) the inequality sign ≤ *is replaced by* ≥, *then the absolute value of the elasticity* $|e_m(I)|$ *is Schur-convex in I on* S_m. *That is, the more non-uniform is the distribution of the total income among the consumers, the larger is the elasticity of the aggregate demand on the considered good by the absolute value.*

<u>Proof.</u> (i) Let $g_m(P,I) = \partial \Phi_m(P,I)/\partial p_m = \sum_{k=1}^{K} \partial \phi_{mk}(P,I_k)/\partial p_m$ be the derivative of the function of aggregate demand on the *m*th good with respect to its price. If conditions (MD.1) and (MD.2) are satisfied, then the following inequalities hold:

$$(I_r - I_s)\left(\frac{\partial \Phi_m(P,I)}{\partial I_r} - \frac{\partial \Phi_m(P,I)}{\partial I_s}\right) =$$

$$(I_r - I_s)\left(\frac{\partial \sum_{k=1}^{K} \phi_{mk}(P,I_k)}{\partial I_r} - \frac{\partial \sum_{k=1}^{K} \phi_{mk}(P,I_k)}{\partial I_s}\right) =$$

$$(I_r - I_s)\left(\frac{\partial \phi_{mr}(P,I_r)}{\partial I_r} - \frac{\partial \phi_{ms}(P,I_s)}{\partial I_s}\right) \geq 0$$

and

$$(I_r - I_s)(\frac{\partial g_m(P,I)}{\partial I_r} - \frac{\partial g_m(P,I)}{\partial I_s}) =$$

$$(I_r - I_s)(\frac{\partial^2 \phi_{mr}(P,I)}{\partial p_m \partial I_r} - \frac{\partial^2 \phi_{ms}(P,I)}{\partial p_m \partial I_s}) \geq 0\,.$$

In addition, from the definition of the functions $\Phi_m(P,I)$ and $g_m(P,I)$ it follows that they are symmetric on the set S_m, that is,

$$\Phi_m(P,\ I_1^{(1)}, I_2^{(1)}, ..., I_K^{(1)}) = \Phi_m(P,\ I_{\pi(1)}^{(1)}, I_{\pi(2)}^{(1)}, ..., I_{\pi(K)}^{(1)}),$$

$$g_m(P,\ I_1^{(1)}, I_2^{(1)}, ..., I_K^{(1)}) = g_m(P,\ I_{\pi(1)}^{(1)}, I_{\pi(2)}^{(1)}, ..., I_{\pi(K)}^{(1)})$$

for all permutations $\pi : \{1,K\} \rightarrow \{1,K\}$ of the set $\{1,K\}$.

Consequently, according to Theorem 3.A.4 in Marshall and Olkin (1979), the functions $\Phi_m(P,I)$ and $g_m(P,I)$ are Schur-convex in I, that is, $I^{(1)} \prec I^{(2)}$ implies $\Phi_m(\mathrm{P}, I^{(1)}) \leq \Phi_\mathrm{m}(\mathrm{P}, I^{(2)})$ and $g_m(P,\ I^{(1)}) \leq g_m(P,\ I^{(2)})$.

Since the function $g_m(P,I)$ is non-positive, from $I^{(1)} \prec I^{(2)}$ it thus follows that

$$\frac{g_m(P,I^{(1)})}{\Phi_m(P,I^{(1)})} \leq \frac{g_m(P,I^{(2)})}{\Phi_m(P,I^{(2)})}$$

or, equivalently,

$$e_m(I^{(1)}) = \frac{\partial \Phi_m(P,I^{(1)})}{\partial p_m} \cdot \frac{p_m}{\Phi_m(P,I^{(1)})} \leq \frac{\partial \Phi_m(P,I^{(2)})}{\partial p_m} \cdot \frac{p_m}{\Phi_m(P,I^{(2)})} = e_m(I^{(2)})\,.$$

That is, $I^{(1)} \prec I^{(2)}$ implies $|em(I^{(2)})| \leq |em(I^{(1)})|$, as claimed.

(ii) If in conditions (MD.1) and (MD.2) the inequality sign $\leq$ is replaced by $\geq$ then the functions $\Phi_m(P,I)$ and $g_m(P,I)$ are Schur-concave in I, that is, $I^{(1)} \prec I^{(2)}$ implies $\Phi_m(\mathrm{P}, I^{(1)}) \geq \Phi_m(\mathrm{P}, I^{(2)})$ and $g_m(P,\ I^{(1)}) \geq g_m(P,\ I^{(2)})$. The rest of the arguments is completely similar to part (i). ■

Example MD.1. Suppose that the function of market demand for good m has the CES form: $\Phi_m(P,I) = \sum_{i=1}^{K} \phi(P, \alpha_{[i]}, I_{[i]})$, where $I_{[1]} \geq I_{[2]} \geq ... \geq I_{[K]}$, $1 > \alpha_{[1]} \geq \alpha_{[2]} \geq ... \geq \alpha_{[K]} > 1/2$, $\phi(P,\alpha,I) = \psi(P,\alpha) I$ and

$$\psi(P,\alpha) = p_m^{-1/(1-\alpha)} \Bigg/ \left(\sum_{j=1}^{M} p_j^{-\alpha/(1-\alpha)} \right)$$

are the factors at the individual CES utility functions (that is, the consumers with a higher income I have a higher elasticity of substitution 1/(1-α)). We have

$$\partial\phi(P,\alpha_r,I_r)/\partial I_r = 1\Bigg/\left(p_m\left(\sum_{j=1}^{M}(p_j/p_m)^{-\alpha_r/(1-\alpha_r)}\right)\right),$$

$$\partial\phi(P,\alpha_r,I_r)/\partial p_m\partial I_r =$$

$$\left(\alpha_r-\sum_{j=1}^{M}(p_j/p_m)^{-\alpha_r/(1-\alpha_r)}\right)\Bigg/\left(p_m^2(1-\alpha_r)\left(\sum_{j=1}^{M}(p_j/p_m)^{-\alpha_r/(1-\alpha_r)}\right)^2\right).$$

Since the function $\sum_{j=1}^{M}(p_j/p_m)^{-\alpha/(1-\alpha)}$ is increasing in $a\in(0,1)$ for $p_j\geq p_m$, $j=1,\ldots,M$, $j\neq m$, we have that $\Phi_m(P,I)$ satisfies conditions (MD.1) if $p_j\geq p_m$, $j=1,\ldots,M$, $j\neq m$. Further, since the function $h(x) = ax^2 - x$ is increasing in x for $x\geq 1/(2\alpha)$, we get that $\Phi_m(P,I)$ satisfies conditions (MD.2) if $p_j/p_m\geq \max_{i=1,\ldots,K}(\frac{M-1}{2\alpha_i-1})^{(1-\alpha_i)/\alpha_i}$ for $j=1,\ldots,M$, $j\neq m$. From part (i) of Theorem MD.1 we obtain that, in this domain, an increase in income inequality leads to a decrease in the absolute value of the market demand elasticity.

Similarly, in the above domain, the market demand function $\Phi_m(P,I)=\sum_{i=1}^{K}\phi(P,\alpha_{(i)})I_{[i]}$, where $\alpha_{(1)}\leq\alpha_{(2)}\leq\ldots\leq\alpha_{(K)}$ and $I_{[1]}\geq I_{[2]}\geq\ldots\geq I_{[K]}$ are ordered in the opposite ways, satisfies conditions (MD.1) and (MD.2) with the inequality signs ≤ replaced by ≥. From part (ii) of Theorem MD.1 we conclude that, in this case, an increase in income inequality leads to an increase in the absolute value of the market demand elasticity.

<u>Example MD.2</u>. Suppose that the function of market demand for good m has the form $\Phi_m(p,I)=\sum_{i=1}^{K}=\phi(p,\alpha_i,\beta_i,I_{[i]})$, where $\phi(p,\alpha_i,\beta_i,I_{[i]})=aI/(I+\beta p)$ is a typical function on goods of first necessity, $\alpha_i,\beta_i>0$, $i=1,\ldots,K,$ are some constants and, as in Example MD.1, $I_{[1]}\geq I_{[2]}\geq\ldots\geq I_{[K]}$. It is not difficult to check that conditions (MD.1) and (MD.2) of part (i) of Theorem MD.1 are satisfied if and only if, for $r\geq s$,

$$\alpha_r\beta_r/(I_{[r]}+\beta_r p)^2\leq\alpha_s\beta_s/(I_{[s]}+\beta_s p)^2, \qquad \text{(MD.3)}$$

$$\alpha_r\beta_r(I_{[r]}-\beta_r p)/(I_{[r]}+\beta_r p)^3\leq\alpha_s\beta_s(I_{[s]}-\beta_s p)/(I_{[s]}+\beta_s p)^3,. \qquad \text{(MD.4)}$$

Let $r \geq s$. Assume that the vector $I=(0,0,\ldots,0)$ belongs to the domain of definition of $\Phi_m(P,I)$**.** Suppose that conditions (MD.1) and (MD.2) of Theorem MD.1 are satisfied. Then from inequalities (MD.3) and (MD.4) for $I=(0,0,\ldots,0)$ it follows that

$$\alpha_r \;/\; \alpha_s \;=\; \beta_r \;/\beta_s. \tag{MD.5}$$

It is easy to see that condition (3) is thus equivalent to $\beta_r \;/\; (I_{[r]} + \beta_r p) \leq \beta_s \;/\; (I_{[s]} + \beta_s p)$ or $I_{[r]} \;/\; \beta_r \geq I_{[s]} \;/\; \beta_s$**.** Since $I_{[r]} \leq I_{[s]}$**,** we conclude that, for conditions (MD.3) and (MD.4) to be satisfied it is necessary that (MD.5) holds for all $r \geq s$ and, in addition, for all $r \geq s$,

$$\beta_r \leq \beta_s, \;\; \alpha_r \leq \alpha_s \;. \tag{MD.6}$$

Suppose that the satiation level for good m is the same for all the consumers, that is, for $p = 0$ and all $r,\ s,$ $\phi(0,\alpha_r,\beta_r,I_r) = \phi(0,\alpha_s,\beta_s,I_s)$. Then from the definition of the individual demand functions ϕ and (MD.5) it follows that $\alpha_r = \alpha_s$ and $\beta_r = \beta_s$ for all $r,\ s.$ Since, as is easy to see, from the above analysis it follows that inequalities (MD.6) are strict for $I_{[r]} < I_{[s]}$ if conditions (MD.1) and (MD.2) are satisfied, we conclude that part (i) of Theorem MD.1 cannot hold.

As above, we get that part (ii) of Theorem MD.1 holds if and only if (MD.3) and (MD.4) are satisfied with the inequality sign $\leq$ replaced by $\geq$. For $I_{[r]} = I_{[s]} = 0$ this implies conditions (MD.5). Assuming that the satiation level for good m is the same for all the consumers, we get that, as above, $\alpha_r = \alpha_s$ and $\beta_r = \beta_s$ for all $r,\ s.$ Thus, it is easy to see that part (i) of Theorem MD.1 holds if and only if, for all $r > s,$

$$(I_{[r]} - \beta p)/(I_{[r]} + \beta p)^3 \geq (I_{[s]} - \beta p)/(I_{[s]} + \beta p)^3, \tag{MD.7}$$

where $\beta = \beta_r = \beta_s$. Similar to Example 1 in Ibragimov and Ibragimov (2007), it is not difficult to check that conditions (MD.7) are satisfied if $I_{[K]} \geq 2\beta p,$ that is if the income levels of all the consumers are not less than $2\beta p$.

APPENDIX WF: Log-linear size-rank regressions and the Weber-Fechner law for income and wealth distributions

As indicated in Section 4, empirical log-linear size-rank regressions (13)-(14) can be interpreted in terms of the Weber-Fechner law that this paper applies to income and wealth data for the first time in the literature.

While Zipf's laws with ζ=1 and, more generally, power laws (1) are inherent to communities and economic and financial markets (e.g., city and firm sizes and financial returns), the Weber-Fechner law is typical for living organisms. The Weber-Fechner law says that the perception will grow in arithmetic progression, when stimuli grow in geometric progression. This law was published in G. Fehner's book "Elements of Psychophysics" in 1859. The Law was discovered in the early 19^{th} century by E. Weber, a German physiologist and psychologist. He studied in detail the link between perception and stimuli when he determined how to change a stimulus for this change to be noticed by a person. It turned out that a ratio of stimulus change (intense) to its initial value is constant:

$$\frac{\Delta S}{S} = b \, ,$$

where S is the stimulus measure, ΔS is the stimulus change/intense, and b is Weber's constant.

Let t=1, …, N, be the rank of household income levels in the *whole* sample under consideration. Let us interpret the rank of income levels as a measure of perception that changes on an arithmetic progression with the step (the difference) equal to 1. Let us also interpret the income level $Z_{(t)}$ as the measure of a stimulus, since ranking has been made according to this parameter. Denote by $\Delta Z_{(t)} = Z_{(t)} - Z_{(t-1)}$, t=2,…, $N,$ the change in the stimulus. Let us suppose that

$$\frac{\Delta Z_{(t)}}{Z_{(t)}} = b = \text{const.}$$

Changing $\Delta Z_{(t)}$ by a differential $dZ_{(t)},$ we have

$$\frac{dZ_{(t)}}{Z_{(t)}} = d \log Z_{(t)} = b = \text{const.}$$

Solving the above differential equation, we obtain relations (8)-(9) that are also equivalent to

$$Z_{(t)} = Aq^{t}, \qquad \text{(WF.1)}$$

where $A = exp(a)$, $q = exp(b)$. The parameter q may be interpreted as the denominator of the geometric progression that corresponds to the change in the “stimulus” $Z_{(t)}$ following the change in the rank in an arithmetic progression.