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Source – Assembly – Sink:

Value Added Flows in the Global Economy

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Abstract

In this paper we provide a method to characterise global value chains and a related decomposition of bilateral gross exports by distinguishing three different stages of the value-added flows: (i) the source of value added, (ii) the final assembly stage of a product, and (iii) the final absorption (sink) of this product. Methodologically this is embedded in a simple framework using matrix algebra allowing for intuitive interpretations of the individual decomposition terms and results. The approach leads to a novel decomposition of bilateral gross export flows and related value-added trade indicators. It is shown how these correspond to existing measures using the property of inverse matrices. Specifically, the paper sheds light on the nature of the double-counting terms discussed in the literature. Finally, the approach outlined is extended by incorporating insights from the hypothetical extraction method. We argue that this is a complementary approach which however can be used to flexibly define the value chains of interest and characterise the respective flows that are considered part of this defined value chain, again carefully differentiating the source, assembly, and sink dimensions.

Keywords: global value chains, decomposition, gross exports, double-counting, hypothetical extraction

JEL classification: F11, F14, F15

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1 Introduction

Since multi-country input-output tables have become available, a number of papers have been published to decompose gross and value-added trade in various dimensions. The seminal contributions (e.g. summarised in Mirodout and Ye, 2017 and Mirodout and Ye, 2021) include Johnson and Noguera (2012), Koopman et al. (2014), Los et al. (2016), and Nagengast and Stehrer (2016). However, there are still questions concerning how exactly to define or calculate the foreign value-added content and double-counting terms in such decompositions (see also Johnson (2017) in this respect). More recent contributions, including Mirodout and Ye (2021), Arto et al. (2019), and Borin and Mancini (2019), tackle some of these challenging questions using different methods and approaches.

This paper contributes to this literature in various aspects. First, we identify three different stages of the value-added flows: (i) the source of the value added, (ii) the final assembly stage of a product, and (iii) the final absorption (sink) of the product. Second, in our methodological framework, we apply simple matrix manipulations (e.g. splitting them into diagonal and off-diagonal elements). This allows us to calculate many of the existing measures already existing in the above-mentioned literature. Importantly, these terms can be easily interpreted in the framework distinguishing the three stages of source, assembly, and sink, and they can be calculated in a straightforward manner. For an easier understanding we explicitly show the appearing terms in the various matrices that allow for an intuitive interpretation along these lines.[1] Third, the methodological approach taken in this paper leads to a novel decomposition of bilateral gross export flows at the country level (the industry dimension is not tackled in this paper). Using some further matrix algebra, specifically applying the property of inverse matrices, we discuss how and in which way this decomposition relates to the gross export decomposition in Koopman et al. (2014) (acknowledging that this decomposition focuses on a country's total exports) and the approach outlined in Los et al. (2016). Fourth, in using this approach, the decomposition of bilateral gross export flows treats the double-counting terms similar to Nagengast and Stehrer (2016) where intermediate goods trade is modelled as a function of final demand or absorption. Comparing the results with the decomposition in Koopman et al. (2014), this paper sheds light on the nature of the double-counting terms appearing there. Specifically, it is argued that the 'double-counted intermediate goods exports originally produced at home' are value-added flows with multiple border crossings for which the source and final assembly country are the same (with the final goods exports being absorbed domestically or exported as a final product). Finally, the approach outlined here is extended by incorporating insights from the hypothetical

[1] This might make this paper also suitable for teaching purposes.

extraction method. We argue that this is a complementary approach which however can be used to flexibly define the value chains of interest and characterise the respective flows that are considered part of this defined value chain, again carefully differentiating the source, assembly, and sink dimensions.

The paper is organised as follows: in Section 2, we explain the characteristics and interpretation of a multi-country input-output table and introduce the matrix notation used throughout the paper. Further, a numerical example including four country groups based on WIOD Release 2016 is presented (including the central gross output (Leontief) and value-added multipliers) on which the results presented throughout the paper are based. In Section 3, we argue that there are two central ways of characterising bilateral value-added trade flows. These characterisations result from two different ways of calculating gross output and the value added in a multi-country Leontief demand-driven model: This leads to a bilateral 'source-sink' and a bilateral 'source-assembly' matrix. The former is closely related to the concept of 'value added trade', whereas the latter is related to the 'value-added content' of trade. Such a distinction allows for neat interpretations and explains that different matrix operations lead to different results. These matrices are central for the subsequent analysis. Based on these analyses, some structural indicators of global value-added flows for the numerical example are presented. The next Section 4 shows various decompositions of the 'source-sink' and 'source-assembly' matrices. These decompositions are based on simple matrix algebra (e.g. splitting matrices into the diagonal and off-diagonal blocks). The resulting terms can be easily interpreted when distinguishing the three stages of production and consumption – source, assembly and sink – with respect to global value-added flows. In addition, a decomposition of the final demand matrix – which in this context can be interpreted as an 'assembly-sink' matrix – is presented. Further, some of the resulting terms and matrices can be aligned with the terms appearing in the approach put forward by Koopman et al. (2014) concerning the decomposition of gross export flows. Section 5 uses these concepts and provides a full bilateral value-added decomposition of gross exports, including both final and intermediary goods; again, these terms allow for an interpretation along the lines of 'source-assembly-sink'. In addition, the relationship to the nine terms in the decomposition (for the country's total gross exports) argued in Koopman et al. (2014) is shown. Using an even more detailed decomposition of the bilateral flows allows one to proof the relationships applying the property of inverse matrices. By doing so we also shed light on the terms that are considered as 'double-counting' in the Koopman et al. (2014) decomposition. In Section 6, we argue that the decomposition presented is a special case of hypothetical extraction method that is presented as an alternative to Koopman et al. (2014) in Los et al. (2016). However, we indicate that the latter approach is more flexible in defining the respective value chains and the characteristics of the chains one wants to study. This is exemplified by splitting out the pure intra-EU flows from the 'source-sink' decomposition presented in Section 4. In the final Section 7, we provide some conclusions and outline further steps. Technical details are presented and explained in the appendix.

2 Multi-country input-output tables and global multipliers

In this section, first, the basic structure of multi-country input-output tables (MC-IOT) and a numerical example using data for the year 2014 (based on the WIOD Release 2016) is presented. For simplification, MC-IOTs are aggregated to four groups of economies (EU-28[2], China, the US, and the Rest-of-the-World RoW). It should be emphasised that all numerical results presented later are calculated at the detailed level of 44 countries and 56 industries and aggregated afterwards only to the total economy levels for these four country groups. Second, the matrix notation used throughout the paper is introduced, and important matrices for the subsequent analysis are defined and interpreted. Analytical examples of matrix calculations are provided for the case of three economies, disregarding the industry dimension. This avoids opaque notation but preserves all the intuitions behind the calculations and results. All results are further presented in full matrix notation, and numerical examples are based on the detailed country- and industry-level data.[3] And, third, the global gross and value-added multiplier matrices are derived and presented because these play an important role in the further analysis.

2.1 Notation and properties

2.1.1 Basic outline of a multi-country input-output table

A multi-country input-output table (MC-IOT) is essentially a tableau that tracks the (nominal) value of physical goods flows across countries and industries, including intra-country and intra-industry flows. These flows - including the industry dimension - are (for three countries) schematically represented as follows:

$$\begin{bmatrix} \mathbf{Z}^{11} & \mathbf{Z}^{12} & \mathbf{Z}^{13} & \mathbf{f}^{11} & \mathbf{f}^{12} & \mathbf{f}^{13} & \mathbf{x}^{1} \\ \mathbf{Z}^{21} & \mathbf{Z}^{22} & \mathbf{Z}^{23} & \mathbf{f}^{21} & \mathbf{f}^{22} & \mathbf{f}^{23} & \mathbf{x}^{2} \\ \mathbf{Z}^{31} & \mathbf{Z}^{32} & \mathbf{Z}^{33} & \mathbf{f}^{31} & \mathbf{f}^{32} & \mathbf{f}^{33} & \mathbf{x}^{3} \\ (\mathbf{w}^{1})' & (\mathbf{w}^{2})' & (\mathbf{w}^{3})' & & & & \\ (\mathbf{x}^{1})' & (\mathbf{x}^{2})' & (\mathbf{x}^{3})' & & & & \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \mathbf{Z} & \mathbf{F} & \mathbf{x} \\ \mathbf{w}' & & \\ \mathbf{x}' & & \end{bmatrix} \tag{2.1}$$

where $\mathbf{Z}^{rc}$ is of dimension $N \times N$ (with N denoting the number of industries), and the vectors $\mathbf{w}^c$, $\mathbf{x}^c$, and $\mathbf{f}^{rc}$ are of dimension $N \times 1$. Consequently, vectors $\mathbf{w}$ and $\mathbf{x}$ are of dimension $NC \times 1$. As one can see, these flows are differentiated by flows of intermediate products (that are further used for the production of other intermediates or final products) $\mathbf{Z}^{rc}$ and final goods (that are absorbed by household consumption,

[2]In 2014, the UK had been member of the EU.

[3]However, it should be stressed that the industry dimension is not exploited further in this paper, and empirical results are presented at the bilateral country level. This means that, for example, we do not distinguish the value-added creation of final absorption by a specific industry in a specific country or the value-added creation in a specific industry due to final demand in another country, and so forth. These industry dimensions are assessed in a companion paper.

investment activities[4], or government expenditures) $\mathbf{f}^{rc}$ either domestically ($r = c$) or abroad ($r \neq c$).[5] In addition, the tables provide information on the value-added creation by industry $\mathbf{w}$ and the gross output produced by industry $\mathbf{x}$ as the sum of intermediary inputs and value added.

For ease of presentation, we aggregate the flows across industries, i.e. $z^{rc} = \mathbf{1}'\mathbf{Z}^{rc}\mathbf{1}$, $f^{rc} = \mathbf{1}'\mathbf{f}^{rc}$, and do the same for value-added and gross output vectors. Here, $\mathbf{1}$ denotes an aggregation vector of ones of the appropriate dimensions. The MC-IOT aggregated to the country level then looks like that presented in equation (2.2).

$$\begin{bmatrix} z^{11} & z^{12} & z^{13} & f^{11} & f^{12} & f^{13} & x^1 \\ z^{21} & z^{22} & z^{23} & f^{21} & f^{22} & f^{23} & x^2 \\ z^{31} & z^{32} & z^{33} & f^{31} & f^{32} & f^{33} & x^3 \\ w^1 & w^2 & w^3 & & & & \\ x^1 & x^2 & x^3 & & & & \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \mathbf{Z} & \mathbf{F} & \mathbf{x} \\ \mathbf{w}' & & \\ \mathbf{x}' & & \end{bmatrix} \tag{2.2}$$

We use this notation for further elaboration and presentation of the decomposition approach in the subsequent chapters. It is again stressed, that all numerical examples provided in the paper are derived from MC-IOT including the full country and industry dimensions.

Variable z^{rc} in equation (2.2) denotes the flows of product (in nominal terms) from country r to c (or intra-country flows from r to r); correspondingly, f^{rc} denotes the value of final goods flows. These can be interpreted as the value of cross-border shipments of goods from country r to c if $r \neq c$, i.e. primary or assembled goods crossing borders, though such goods might include parts and components that have already crossed borders multiple times. A flow from r to c with $r \neq c$ thus denotes the value of exports of country r to c and – by definition – the value of imports of country c from r. Further, production of a good requires the use of primary inputs (e.g. labour, capital, or land). These factors earn their income denoted by w^c, which constitutes a country's gross domestic product (GDP) as the sum of the income of the various production factors (e.g. wages and profits).[6] The value of the gross output a country produces, denoted by x^c, is the summation of the value of domestically produced and imported intermediary products and the income of the primary factors, i.e. $x^c = \sum_r z^{rc} + w^c$. Final goods demand f^{rc} denotes the value of consumption (or absorption) in country c. The products have either been finally assembled in this country $r = c$ or imported from the country of final assembly $r \neq c$.

Such a multi-country input-output table satisfies various constraints according to National Accounting identities: First, the value of gross output x^c (the column sum already discussed above) is equal to the value the country delivers to its own economy and other countries (i.e. exports) as intermediary or final

[4]Changes in inventories are treated as part of investment.

[5]Throughout the paper, superscript r is used for the 'row-country', and superscript c is used for the 'column-country'. Further, the convention throughout this paper is that column vectors are represented as lowercase characters in bold font; the corresponding row vectors (i.e. their transpose) are indicated by $'$. Matrices are represented using bold capital letters.

[6]These would also include taxes on production, mixed income, etc. that are not considered here separately.

products (the row sum). These deliveries are the row sum and satisfy $x^c = \sum_c (z^{rc} + f^{rc})$. Second, the gross domestic product of all countries together – or the world GDP $\sum_c w^c$ – is equal to the value of final goods demand in the world $\sum_{r,c} f^{rc}$. Note that this has to hold at the world level, but it does not necessarily have to hold at the level of individual countries that can make trade surpluses or deficits, which will be discussed later in more detail. However, the trade balances across all countries have to sum up to zero by definition.

2.1.2 Empirical example

A numerical example – used throughout the paper – of such a multi-country input-output table is shown in Table 2.1. This is derived from the WIOD (Release 2016) input-output table documented in Timmer et al. (2015) for year 2014 and aggregated over industries and countries and distinguishes the four country groups mentioned above: EU-28, US, China and the Rest-of-the-World (RoW).

Table 2.1: Aggregated multi-country input-output table, 2014

	Intermediates				Final demand				Total final	
	EU-28	China	USA	RoW	EU-28	China	USA	RoW	demand	Sum*
EU-28	15,252	139	273	1,504	15,756	138	206	1,093	17,192	34,361
China	185	19,972	130	897	181	9,348	217	815	10,561	31,745
USA	324	66	12,164	871	130	46	16,880	491	17,546	30,971
RoW	1,256	1,169	987	30,362	520	285	595	28,748	30,148	63,920
Value added	17,345	10,399	17,417	30,287	16,586	9,816	17,897	31,147	75,447	75,447
Gross output	34,361	31,745	30,971	63,920						160,997

Note: Values in bn USD; *not including column 'Total final demand'
Source: WIOD Release 2016; own calculations.

These aggregates therefore include intra-country flows and particularly inter-country flows within the regions EU-28 and RoW. The properties mentioned above can easily be verified, i.e. the column sum equals the row sums and world GDP equals world final demand expenditures.[7] It is interesting to note that the EU-28, China and the US are of more or less the same size in terms of gross output, whereas in terms of value added, China accounts for two-thirds compared to the other two; RoW is about twice as big as the other countries individually in terms of gross output. The ratio of value added to gross output is about 0.5 for EU-28, slightly higher with 0.56 for the US, 0.3 for China, and 0.48 for RoW. Further interesting insights are discussed throughout the paper.

2.1.3 Useful matrix splits and aggregates

For the subsequent analysis provided in this paper, it is useful to represent the multi-country input output table in matrix notation as shown in equations (2.1) or (2.2). For the following analysis, the matrices of

[7] In some cases, small rounding errors might be prevalent.

intermediary and final good flows are split into its diagonal (indicated by ˆ) and off-diagonal elements (indicated by ˜). When considering the MC-IOT with the industry dimension, as in equation (2.1), the diagonal elements would be the respective block-diagonal matrices, and for final demand, the respective vectors are $\mathbf{f}^{rr}$; for simplicity, these are then also referred to as the 'block-diagonal' elements of matrix $\mathbf{F}$. Formally, these matrices are split into these two parts and – for the example of three economies – given by

$$\mathbf{Z} = \begin{pmatrix} z^{11} & z^{12} & z^{13} \\ z^{21} & z^{22} & z^{23} \\ z^{31} & z^{32} & z^{33} \end{pmatrix} = \hat{\mathbf{Z}} + \tilde{\mathbf{Z}} = \begin{pmatrix} z^{11} & 0 & 0 \\ 0 & z^{22} & 0 \\ 0 & 0 & z^{33} \end{pmatrix} + \begin{pmatrix} 0 & z^{12} & z^{13} \\ z^{21} & 0 & z^{23} \\ z^{31} & z^{32} & 0 \end{pmatrix}$$

and

$$\mathbf{F} = \begin{pmatrix} f^{11} & f^{12} & f^{13} \\ f^{21} & f^{22} & f^{23} \\ f^{31} & f^{32} & f^{33} \end{pmatrix} = \hat{\mathbf{F}} + \tilde{\mathbf{F}} = \begin{pmatrix} f^{11} & 0 & 0 \\ 0 & f^{22} & 0 \\ 0 & 0 & f^{33} \end{pmatrix} + \begin{pmatrix} 0 & f^{12} & f^{13} \\ f^{21} & 0 & f^{23} \\ f^{31} & f^{32} & 0 \end{pmatrix}$$

Note that the off-diagonal parts of the matrix, i.e $\tilde{\mathbf{Z}}$ and $\tilde{\mathbf{F}}$, represent the cross-border trade flows of intermediary and final goods, respectively. For further use, the row sum of the final goods demand matrix $\mathbf{F}$ is given by

$$\mathbf{f} = \mathbf{F} \cdot \mathbf{1} = \begin{pmatrix} f^{11} & f^{12} & f^{13} \\ f^{21} & f^{22} & f^{23} \\ f^{31} & f^{32} & f^{33} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} f^{11} + f^{12} + f^{13} \\ f^{21} + f^{22} + f^{23} \\ f^{31} + f^{32} + f^{33} \end{pmatrix} = \begin{pmatrix} f^{1*} \\ f^{2*} \\ f^{3*} \end{pmatrix}$$

Each element of this vector shows the domestic and foreign demand that country r can attract on the products it finally assembles.[8]

2.1.4 Gross exports and trade balances

Having defined these matrices, gross exports are given by the row sum of the off-diagonal elements of the transactions matrix (flow of intermediary products) aggregated over using industries and the final goods

[8] In case the industry dimension is considered, vector $\mathbf{f}$ would be of dimension $CN \times 1$.

matrix that has to be aggregated over industries[9] arriving at

$$\tilde{\mathbf{E}} = \begin{pmatrix} 0 & e^{12} & e^{13} \\ e^{21} & 0 & e^{23} \\ e^{31} & e^{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & f^{12} & f^{13} \\ f^{21} & 0 & f^{23} \\ f^{31} & f^{32} & 0 \end{pmatrix} + \begin{pmatrix} 0 & z^{12} & z^{13} \\ z^{21} & 0 & z^{23} \\ z^{31} & z^{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^{12}+f^{12} & z^{13}+f^{13} \\ z^{21}+f^{21} & 0 & z^{23}+f^{23} \\ z^{31}+f^{31} & z^{32}+f^{32} & 0 \end{pmatrix}$$

The bilateral (gross) trade balances are then - after aggregating over industries - given by

$$\tilde{\mathbf{E}} - \tilde{\mathbf{E}}' = \begin{pmatrix} 0 & e^{12}-e^{21} & e^{13}-e^{31} \\ e^{21}-e^{12} & 0 & e^{23}-e^{32} \\ e^{31}-e^{13} & e^{32}-e^{23} & 0 \end{pmatrix}$$

These bilateral gross trade flows in intermediary and final goods as well as total trade together with the gross trade balances are shown in Table 2.2 for our empirical example.[10]

Table 2.2: Aggregated gross trade flows and trade balances, 2014

Importer \ Exporter	EU-28	China	USA	RoW	Exports	EU-28	China	USA	RoW	Exports
	Intermediate goods					Final goods				
EU-28	2,484	139	273	1,504	4,401	1,387	138	206	1,093	2,823
China	185	0	130	897	1,212	181	0	217	815	1,213
USA	324	66	0	871	1,261	130	46	0	491	666
RoW	1,256	1,169	987	3,053	6,465	520	285	595	1,208	2,607
Imports	4,249	1,374	1,390	6,325	13,339	2,217	468	1,018	3,607	7,310
	Total trade					Trade balances				
EU-28	3,871	277	479	2,597	7,224	0	-89	26	822	759
China	366	0	347	1,712	2,425	89	0	235	258	583
USA	453	112	0	1,362	1,927	-26	-235	0	-219	-481
RoW	1,775	1,454	1,581	4,261	9,072	-822	-258	219	0	-861
Imports	6,465	1,843	2,408	9,933	20,649	-759	-583	481	861	0

Note: Values in bn USD; includes intra-regional trade.
Source: WIOD Release 2016; own calculations.

Note that the diagonal cells of the trade flows are occupied for EU-28 and RoW because these calculations include inter-country flows within these regions (though excluding intra-country flows). For example, intra EU-28 trade amounts to 3,871 bn USD. Focusing on the trade balances, the EU-28, for example, runs a trade surplus of 759 bn USD, which mostly stems from trade with RoW (822 bn USD) and, to a lesser extent, with the US (26 bn USD). However, the EU-28 has a bilateral trade deficit with

[9]Formally, when considering the industry dimension, this requires one to calculate $\tilde{\mathbf{Z}}_a = \tilde{\mathbf{Z}}(\mathbf{I} \otimes \mathbf{1}')$ (where $\otimes$ denotes the Kronecker product, $\mathbf{I}$ is the identity matrix with dimension $C \times C$, and $\mathbf{1}$ is a vector of dimension $N \times 1$), i.e. aggregating the transactions matrix $\mathbf{Z}$ row-wise over the using industries of each country. This results in a $C \cdot N \times C$ matrix compatible with the dimensionality of $\tilde{\mathbf{F}}$. Then, to calculate a country's exports, these have to be aggregated across industries for each country (pre-multiplying with $\mathbf{I} \otimes \mathbf{1}$) resulting in matrices of dimension $C \times C$.

[10]One could further calculate trade balances for intermediates and final goods separately.

China of 89 bn USD. Analogous interpretations hold for the other countries. Note, that the trade balance of the world, and also for intra-regional trade in the case of EU-28 and RoW, is zero by definition.

2.2 Global gross and value added multipliers

For input-output analysis and value-added trade indicators, the coefficient matrix of intermediary inputs (intermediary use per unit of gross output) and the Leontief inverse are central tools. The input-output coefficients are defined as intermediary input flows relative to gross output and denoted by $a^{rc} = z^{rc}/x^c$, or in matrix notation $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$, where $\hat{\mathbf{x}}$ denotes the diagonalized vector of gross output levels. In detailed notation this is given by

$$\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} = \begin{pmatrix} z^{11} & z^{12} & z^{13} \\ z^{21} & z^{22} & z^{23} \\ z^{31} & z^{32} & z^{33} \end{pmatrix} \begin{pmatrix} 1/x^1 & 0 & 0 \\ 0 & 1/x^2 & 0 \\ 0 & 0 & 1/x^3 \end{pmatrix} = \begin{pmatrix} a^{11} & a^{12} & a^{13} \\ a^{21} & a^{22} & a^{23} \\ a^{31} & a^{32} & a^{33} \end{pmatrix}$$

The left panel in Table 2.3 shows the resulting input-output coefficients a^{rc} stemming from the numerical example.[11] These numbers can also be interpreted as cost shares. For example, 44% of the value of gross output in the EU-28 is due to intermediary inputs from the EU-28, 0.5% from China, 0.9% from the US, and 3.7% from RoW. In total, the share of intermediary inputs is 49.5%. Primary factor income, i.e. value added, accounts for the remaining part. Analogous interpretations hold for the other countries.

Having derived the coefficient matrix $\mathbf{A}$, the Leontief inverse indicating the directly and indirectly needed gross output for the production of a unit of a final good is given by

$$\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1} = \begin{pmatrix} l^{11} & l^{12} & l^{13} \\ l^{21} & l^{22} & l^{23} \\ l^{31} & l^{32} & l^{33} \end{pmatrix}$$

where $\mathbf{I}$ denotes the identity matrix (of appropriate dimension). This is referred to as the 'global' Leontief matrix as derived from the 'global' coefficients matrix. The column sum of the Leontief inverse matrix is commonly known as (global) gross-output multipliers. This Leontief matrix and the corresponding gross output multipliers resulting from the numerical example are presented in the middle panel of Table 2.3.[12] The interpretation is as follows: To produce one unit (i.e. 1 bn USD) more of demand for EU-28 final products (consumed in the EU-28 or exported) needs the production of gross output of more than 2.1 bn USD, of which 1.875 bn USD are created in the EU-28, 0.038 bn USD in China, 0.034 bn USD in the US and 0.156 bn USD in RoW. Analogous interpretations hold for the other countries.

[11] In detail, the global coefficients matrix is calculated by country and industry. Each column is aggregated over country groups and industry by simply summing up the coefficients. Row-wise aggregation is performed using gross output weights by industry and country group.

[12] These are as well calculated by country and industry and aggregated in the same way as the coefficients matrix.

Table 2.3: Coefficient matrix and multipliers

	Input coefficients				Gross output multiplier				Value added multiplier			
	EU-28	China	USA	RoW	EU-28	China	USA	RoW	EU-28	China	USA	RoW
EU-28	0.444	0.004	0.009	0.024	1.873	0.038	0.037	0.098	0.905	0.016	0.016	0.041
China	0.005	0.629	0.004	0.014	0.038	2.729	0.029	0.087	0.011	0.877	0.008	0.025
USA	0.009	0.002	0.393	0.014	0.034	0.017	1.678	0.050	0.018	0.008	0.925	0.025
RoW	0.037	0.037	0.032	0.475	0.156	0.258	0.123	2.029	0.066	0.099	0.052	0.909
Total	0.495	0.672	0.438	0.526	2.102	3.042	1.867	2.265	1.000	1.000	1.000	1.000

Source: WIOD Release 2016; own calculations.

The gross output multipliers can be converted into value-added multipliers. These show the amount of value added produced for the production of a unit (i.e. 1 bn USD) of a final product. For doing so, the value-added coefficients are defined as the share of value added in gross output, in matrix notation $\mathbf{v} = \hat{\mathbf{x}}^{-1}\mathbf{w}$, i.e. the inverse of the diagonalized gross output vector times the value added levels.[13] The value-added multiplier matrix is then given by multiplying the diagonalized vector of value-added coefficients with the Leontief inverse matrix and is denoted by

$$\mathbf{B} = \hat{\mathbf{v}}\mathbf{L} = \begin{pmatrix} b^{11} & b^{12} & b^{13} \\ b^{21} & b^{22} & b^{23} \\ b^{31} & b^{32} & b^{33} \end{pmatrix} = \begin{pmatrix} v^1 & 0 & 0 \\ 0 & v^2 & 0 \\ 0 & 0 & v^3 \end{pmatrix} \begin{pmatrix} l^{11} & l^{12} & l^{13} \\ l^{21} & l^{22} & l^{23} \\ l^{31} & l^{32} & l^{33} \end{pmatrix} = \begin{pmatrix} v^1 l^{11} & v^1 l^{12} & v^1 l^{13} \\ v^2 l^{21} & v^2 l^{22} & v^2 l^{23} \\ v^3 l^{31} & v^3 l^{32} & v^3 l^{33} \end{pmatrix}$$

The column sum of the value-added multiplier matrix are the 'value-added multipliers' and are – by definition – equal to one as indicated in the right panel of Table 2.3.[14] This results from the fact that the total value added produced in the world is equal the total value of final demand (being one of the fundamental properties of MC-IOT), and is also reflected in the fact that the cost shares of intermediary inputs and value added in the gross output add up to 1 by definition. In the numerical example, an increase of 1 bn USD of final demand on EU-28 products generates value added created in the world of 1 bn USD, of which 0.905 bn USD are created in the EU-28, 0.011 in China, 0.018 in the US and 0.066 bn USD in RoW. Analogous interpretations hold for the other three countries.

[13] Note that by definition, the value-added coefficients are also one minus the sum of the cost shares of intermediates in gross output (i.e. the intermediary input coefficients), thus $v^c = 1 - \sum_r a^{rc}$, or $\mathbf{v}' = \mathbf{1}' - \mathbf{1}'\mathbf{A} = \mathbf{1}'(\mathbf{I} - \mathbf{A})$.

[14] Again, these are calculated from the country- and industry-level Leontief inverse and value-added coefficients and aggregated in the same way as the coefficients of the Leontief matrix.

3 Source, assembly, and sink

Based on the methodological outline presented in the previous section, we provide two versions of the demand-driven Leontief model in an international context which allows us to interpret the value-added flows in the global economy from (i) the origin of value added (source) to the absorption of value added (sink) or (ii) from the source of value added to the stage of the assembly of the final product.[15] Further, in this context, the final demand matrix $\mathbf{F}$ can be interpreted in the way that a typical element f^{rc} indicates the country of final assembly r and the country of absorption (sink) c, thus as an 'assembly-sink' matrix. This is not further explored as no additional calculations are required. The analytical statements are accompanied by numerical examples. For completeness, the tables also include the corresponding gross output values for sake of completeness. The final subsection of this chapter provides some empirical insights based on these calculations.

3.1 The 'demand driven international Leontief model'

In the standard Leontief demand-driven model, the gross output multiplier matrix (Leontief matrix) is multiplied by a vector of final demand, which results in the vector of gross output, i.e. $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f} = \mathbf{Lf}$. Pre-multiplying this expression with the diagonalized vector of value-added coefficients results in a vector of value-added levels, i.e. $\hat{\mathbf{v}}\mathbf{x} = \hat{\mathbf{v}}\mathbf{Lf} = \mathbf{Bf} = \mathbf{w}$ (see Section 2). For the following discussion, it is important to notice that in a multi-country setting, the gross output and the value added vector can be calculated in two ways. Using the notation introduced in Section 2 one can first write the demand-driven Leontief model as

$$\mathbf{x} = \mathbf{L} \cdot (\mathbf{F} \cdot \mathbf{1}) = \mathbf{L} \cdot \mathbf{f} = [\mathbf{L} \cdot \hat{\mathbf{f}}] \cdot \mathbf{1} \qquad \text{and} \qquad \mathbf{w} = \mathbf{B} \cdot \mathbf{f} = \mathbf{B} \cdot (\mathbf{F} \cdot \mathbf{1}) = [\mathbf{B} \cdot \hat{\mathbf{f}}] \cdot \mathbf{1} \tag{3.1}$$

which closely corresponds to the standard demand-driven Leontief model (as the Leontief inverse is post-multiplied with a corresponding vector of final demand). Second, the same gross output and value-added vectors are achieved by first multiplying the Leontief inverse and final demand matrix and then building the row sums, i.e.

$$\mathbf{x} = [\mathbf{L} \cdot \mathbf{F}] \cdot \mathbf{1} \qquad \text{and} \qquad \mathbf{w} = [\mathbf{B} \cdot \mathbf{F}] \cdot \mathbf{1} \tag{3.2}$$

The matrices in brackets in both expressions are of interest in the context of this paper because these lean towards the different interpretations with respect to value-added flows. In this context it is important to note that these matrices are not equal, though the row sums of these expressions are equal as resulting

[15] The first version leads to an interpretation of 'trade in value added', whereas the second one leads to an interpretation of the 'value added in trade' as introduced in Stehrer (2012).

in the same gross output or value-added vectors, respectively.[16] Both versions, however, yield important insights in the gross output and value-added generation and global flows, which becomes important for the interpretation, calculations and decomposition of gross and value-added trade flows. For reasons that will become clear in the next two subsections, the above expressions are referred to as 'source-assembly' and 'source-sink' matrices.

3.2 Source and sink

Starting with the source-sink matrix, this formally requires one to multiply the value-added multiplier matrix with the final demand matrix.[17] For three countries, the resulting expression looks like

$$\mathbf{T} = \mathbf{B} \cdot \mathbf{F} = \begin{pmatrix} b^{11}f^{11} + b^{12}f^{21} + b^{13}f^{31} & b^{11}f^{12} + b^{12}f^{22} + b^{13}f^{32} & b^{11}f^{13} + b^{12}f^{23} + b^{13}f^{33} \\ b^{21}f^{11} + b^{22}f^{21} + b^{23}f^{31} & b^{21}f^{12} + b^{22}f^{22} + b^{23}f^{32} & b^{21}f^{13} + b^{22}f^{23} + b^{23}f^{33} \\ b^{31}f^{11} + b^{32}f^{21} + b^{33}f^{31} & b^{31}f^{12} + b^{32}f^{22} + b^{33}f^{32} & b^{31}f^{13} + b^{32}f^{23} + b^{33}f^{33} \end{pmatrix} \quad (3.3)$$

A specific cell, $\sum_s b^{rs} f^{sc}$, can be interpreted as the value added generated in a row ('source') country r to satisfy a column ('sink') country's c demand for final products. This explains why we refer to this matrix as a 'source-sink' matrix. The row sums are equal to a country's total value added (as already outlined above). The column sums of are equal to the country's final demand (either produced domestically or imported), i.e. $\mathbf{1'F}$.[18] The diagonal cells in the matrix indicate the cases where the source-country also equals the sink-country (i.e. $r = c$), whereas the off-diagonal elements indicate cases where the source-country is different from the sink-country (i.e. $r \neq c$). Therefore, disregarding the diagonal entries results in the value added generated in one country but finally absorbed in another country, thus indicating the bilateral 'value added trade' (or 'trade in value added'). Using the notation from the previous section, matrix $\mathbf{T}$ with the diagonal elements set to zero can be written as $\tilde{\mathbf{T}}$.[19]

The resulting figures for our numerical example are presented in Table 3.1.[20] As one can see in the upper part of this table, the row sums equal the country's value added and gross output figures in Table 2.1, and the column sums equal the country's total final demand.[21] Interpreting the figures from the perspective of the EU-28, for example, 14,648 bn USD of value added is generated in the EU-28 due to EU-28 demand on final products assembled domestically or imported from abroad. Analogous interpretations hold for the other countries when considering the diagonal elements. Going along the

[16] Formally, $[\mathbf{L} \cdot \hat{\mathbf{f}}] \neq [\mathbf{L} \cdot \mathbf{F}]$ and $[\mathbf{B} \cdot \hat{\mathbf{f}}] \neq [\mathbf{B} \cdot \mathbf{F}]$, however $[\mathbf{L} \cdot \hat{\mathbf{f}}]\mathbf{1} = [\mathbf{L} \cdot \mathbf{F}]\mathbf{1}$ and $[\mathbf{B} \cdot \hat{\mathbf{f}}]\mathbf{1} = [\mathbf{B} \cdot \mathbf{F}]\mathbf{1}$.

[17] The dimension of matrix $\mathbf{B}$ is $NC \times NC$, and the dimension of matrix $\mathbf{F}$ is $NC \times C$. Thus this results in a matrix of dimension $NC \times C$ that after aggregation over industries is of dimension $C \times C$.

[18] This follows from the fact that – by definition – the column sums of the value-added multiplier matrix are given by $\mathbf{1'B} = \mathbf{1'}$, thus $\mathbf{1'BF} = \mathbf{1'F}$.

[19] Note that this is different from calculating $\mathbf{B}\tilde{\mathbf{F}}$, i.e. disregarding domestic demand on domestically assembled products, which will become clear in Section 4.

[20] Again, all calculations are performed at the detailed country and industry level and are then summed up over industries and the respective country groups for presentational purposes.

[21] Small deviations occur due to rounding errors.

Table 3.1: Source and sink

Source \ Sink	Gross output EU-28	China	USA	RoW	Total	Value added EU-28	China	USA	RoW	Total
					Total					
EU-28	27,932	647	1,062	4,719	34,361	14,648	270	444	1,983	17,345
China	1,132	24,830	1,221	4,562	31,745	312	8,441	327	1,319	10,399
USA	677	258	27,810	2,225	30,971	346	122	15,860	1,089	17,417
RoW	3,140	2,604	3,165	55,012	63,920	1,278	983	1,263	26,762	30,287
Total	32,882	28,339	33,259	66,517	160,997	16,583	9,816	17,894	31,154	75,447
					Value added exports* $\tilde{\mathbf{T}}$					
EU-28	5,252	647	1,062	4,719	11,680	2,132	270	444	1,983	4,829
China	1,132	0	1,221	4,562	6,915	312	0	327	1,319	1,958
USA	677	258	0	2,225	3,161	346	122	0	1,089	1,557
RoW	3,140	2,604	3,165	7,303	16,211	1,278	983	1,263	2,986	6,510
Total	10,202	3,509	5,448	18,808	37,968	4,068	1,375	2,034	7,377	14,854

Note: Values in bn USD; *including intra-regional trade for EU-28 and RoW
Source: WIOD Release 2016; own calculations.

row, the next cell (270 bn USD) denotes value added generated in the EU-28, which is finally absorbed in China (i.e. consuming products, which are either finally assembled in China or any other country (including the EU-28) and imported). Consequently, this constitutes value-added exports of the EU-28 to China (or, analogously, Chinese value-added imports from the EU-28). Going down the column of the EU-28, the figures indicate that 312 bn USD is value added generated in China, which is finally absorbed in the EU-28, i.e. value-added exports of China to the EU-28 (or value added imports of the EU-28 from China). Analogous interpretations hold for all other off-diagonal cells.

The lower panel in this table presents the value-added trade matrix, which consists of the off-diagonal elements, i.e. $\tilde{\mathbf{T}}$.[22] As calculations are performed at the detailed country- and industry-level aggregation to the country group level presented in the table, they still include inter-country flows for EU-28 and RoW (e.g. value-added flows from Austria to France), but not the intra-country flows. Therefore, for these countries, the numbers at the diagonal represent value-added exports within the countries in the respective groups. For example, 2,132 bn USD of value added is generated in the EU-28 and finally absorbed in the EU-28, excluding the cases where source- and sink-country are the same.

3.3 Source and assembly

The second method is to multiply the Leontief matrix $\mathbf{L}$ or the value-added multiplier matrix $\mathbf{B}$ with the diagonalized vector of final demand $\mathbf{f}$ that results in equation (3.4).[23] In this case, a specific cell, $\sum_c b^{rc} f^{cs}$, can be interpreted as the value added generated in the row ('source') country r and embodied

[22] This corresponds to the 'trade in value added' concept introduced in Johnson and Noguera (2012) and Stehrer (2012).

[23] Matrix $\mathbf{B}$ and the matrix $\hat{\mathbf{f}}$ have dimension $NC \times NC$, thus the resulting matrix $\mathbf{B}$ also has dimension $NC \times NC$. These can be added across the industry dimensions resulting in a country-level matrix with dimension $C \times C$.

in the final product in the assembly country c. Therefore, this matrix is referred to as 'source-assembly' matrix. This final product is then either consumed domestically $c = s$ or exported $c \neq s$.

$$\mathbf{C} = \mathbf{B} \cdot \hat{\mathbf{f}} = \begin{pmatrix} b^{11}(f^{11} + f^{12} + f^{13}) & b^{12}(f^{21} + f^{22} + f^{23}) & b^{13}(f^{31} + f^{32} + f^{33}) \\ b^{21}(f^{11} + f^{12} + f^{13}) & b^{22}(f^{21} + f^{22} + f^{23}) & b^{23}(f^{31} + f^{32} + f^{33}) \\ b^{31}(f^{11} + f^{12} + f^{13}) & b^{32}(f^{21} + f^{22} + f^{23}) & b^{33}(f^{31} + f^{32} + f^{33}) \end{pmatrix} \tag{3.4}$$

As before, the row sums add up to each country's total value added. Conversely, the column sums indicate the value added embodied in the products finally assembled in the respective country c. This equals the value of final demand a country is able to attract and therefore equals the row sum of the final demand matrix $\mathbf{F} \cdot \mathbf{1} = \mathbf{f}$.[24] The diagonal cells indicate the domestic content, i.e. $r = c$, whereas the off-diagonal cells indicate the foreign content as $r \neq c$. Finally, when disregarding the domestically assembled and absorbed products, i.e. the terms including f^{cc} in each cell, one gets the domestic and foreign contents of a country's final good exports (see below for a technical outline). Table 3.2 shows the results from the numerical example; the left panel of this table presents the corresponding gross output values $\mathbf{L}\hat{\mathbf{f}}$.[25]

Table 3.2: Source and assembly

Source \ Assembly	Gross output: EU-28	China	USA	RoW	Total	Value added: EU-28	China	USA	RoW	Total
Total final demand										
EU-28	30,756	379	590	2,635	34,361	15,826	158	247	1,113	17,345
China	593	28,404	477	2,272	31,745	165	9,440	126	667	10,399
USA	498	174	28,883	1,416	30,971	259	84	16,384	690	17,417
RoW	2,237	2,337	1,882	57,464	63,920	940	878	786	27,683	30,287
Total	34,083	31,294	31,832	63,788	160,997	17,190	10,561	17,544	30,152	75,447
Final goods exports* $\mathbf{B}(\widehat{\tilde{\mathbf{F}}\mathbf{1}})$										
EU-28	5,877	60	41	374	6,352	2,400	25	17	156	2,598
China	201	3,650	38	394	4,283	55	1,020	10	110	1,195
USA	170	27	1,207	247	1,651	88	13	586	117	804
RoW	690	438	134	5,778	7,040	279	155	54	2,221	2,710
Total	6,937	4,175	1,420	6,795	19,327	2,822	1,213	666	2,605	7,307

Note: Values in bn USD; including intra-regional trade for EU-28 and RoW.
Source: WIOD Release 2016; own calculations.

First, as mentioned already, the row sums up to the country's total value added and gross output levels, respectively. Second, the columns add up to demand a country can attract for finally assembled products, i.e. the row sum of the final demand matrix (see Table 2.1).[26] For example, one can see that EU-28 attracts 17,190 bn USD of final demand. This value is composed of value added generated in the

[24] Formally, this again results from $\mathbf{1}'\mathbf{B} = \mathbf{1}'$.
[25] As before, calculations are performed at the detailed country and industry level. Results are then aggregated over industries and summed up over country groups.
[26] Again, some small rounding errors occur.

EU-28 itself (15,826 bn USD) and the other countries, e.g. 165 bn USD generated in China, which enters final assembly (but not necessarily absorption) in the EU-28. Analogous interpretations hold for the other countries. Focusing on the exported final products only, one has to disregard domestic demand on the domestically finally assembled products, i.e. $\hat{\mathbf{F}}$ and calculate $\mathbf{B}(\widehat{\hat{\mathbf{F}}\mathbf{1}})$. This is shown in the lower part of Table 3.2 that therefore indicates the value added embodied in a country's final demand exports.[27] Accordingly, the EU-28 exports a value of 2,822 bn USD of final goods embodying 2,400 bn USD domestic (i.e. EU-28) value added, 55 bn USD value added originating in China, 88 bn USD from the US, and 279 bn USD from RoW.

3.4 Structural indicators of value-added flows

Using the results of these two approaches, some descriptive indicators can be calculated. For some important examples, one can, first, easily calculate the bilateral trade balances in terms of value added; second, one can calculate the 'value added intensity of bilateral trade flows'; and, third, of course, the respective country shares concerning the various value-added flows can be calculated.

3.4.1 Bilateral value added trade balances

Above, the value-added exports matrix $\tilde{\mathbf{T}}$ has already been discussed. The bilateral value-added trade balances are then calculated as $\tilde{\mathbf{T}} - \tilde{\mathbf{T}}'$.[28] The figures are reported in Table 3.3. These figures can be compared to the bilateral trade balances in gross terms reported in Table 2.2. As one can see, overall

Table 3.3: Bilateral and total value added trade balances

Source \ Sink	EU-28	China	USA	RoW	Total
EU-28	0	-42	98	706	762
China	42	0	205	335	583
USA	-98	-205	0	-174	-477
RoW	-706	-335	174	0	-867
Total	-762	-583	477	867	0

Note: Values in bn USD.
Source: WIOD Release 2016; own calculations.

net trade positions remain the same.[29] The reason is simply that a country's trade balance is just the difference between the value added produced minus final consumption.[30] However, bilateral trade balances in value-added terms differ from those in gross terms. For example, the EU-28 shows a trade

[27] These calculations thus show the value-added content of exported products, which also can be referred to as 'value added in trade' (VAiT; see e.g. Stehrer, 2012), which in this case are applied to final goods exports only. More details are provided in Section 4.

[28] Bilateral value-added trade balances can also be calculated using matrix $\mathbf{T}$ in an analogous way as intra-country flows drop out.

[29] Small differences are due to rounding errors.

[30] See Stehrer (2012) for a formal treatment.

deficit of 89 bn USD with China in gross exports, whereas the bilateral trade balances in value-added terms is about only half this, with 42 bn USD. The trade surplus in value-added terms with the US is 98 bn USD, compared to a trade surplus in gross terms of 26 bn USD, and with RoW, the corresponding numbers are 706 bn USD in value-added terms compared to 822 bn USD in gross terms. Another example is the US, which runs a trade deficit of 235 bn USD against China in gross terms; this deficit is reduced to 205 bn USD in value-added terms.[31]

3.4.2 Value-added intensity of bilateral trade

Second, the value-added trade matrix **BF** can be compared with the gross output trade matrix **LF**.[32] A simple method is to calculate the ratio of value-added exports to the corresponding gross output figures, which are reported in Table 3.4 as an indicative example. The last column shows the value added to gross

Table 3.4: Bilateral and total value-added trade intensities

Source \ Sink	EU-28	China	USA	RoW	Total
EU-28	0.524	0.417	0.418	0.420	0.505
China	0.276	0.340	0.268	0.289	0.328
USA	0.511	0.471	0.570	0.490	0.562
RoW	0.407	0.378	0.399	0.486	0.474
Total	0.504	0.346	0.538	0.468	0.469

Source: WIOD Release 2016; own calculations.

output ratio for the total economy. For example, in the EU-28, 50.5% of gross output is value added with the remaining share being intermediate inputs. This share is much lower in China with 32.8%, slightly lower for the RoW with 47.4%, and higher in the US with 56.2%. The value-added shares of the intra-country flows (in the diagonal cells) are in all cases higher than those for value-added exports, reflecting the higher share of services (which are usually characterised by higher value-added shares and lower trade shares). The value-added ratios for export flows (the off-diagonal cells) are about five to ten percentage points lower than the overall shares. For example, it is interesting to note that there is a substantial difference between the bilateral value added exports between the EU-28 and the US. The trade flows from the US to the EU-28 show a value-added ratio of 51.1%, which is around five percentage points lower than the overall ratio in the US of 56.2%. The trade flows from the EU-28 to the US are characterised by a ratio of 41.8%, which is eight percentage points lower than the EU-28 overall ratio (50.5%).

[31] For a detailed discussion and decomposition of trade balances in a similar framework, see Nagengast and Stehrer (2016).
[32] This should not be confused with the matrix of bilateral gross exports.

3.4.3 Structure of 'source-sink' and the 'source-assembly' flows

Finally, the next two tables show the structure of the 'source-sink' (value-added trade) matrix and the 'source-assembly' (value-added content of trade) matrix derived above.

Structure of value-added trade From the upper panel of Table 3.5 one can see that 84.4% of value added produced in the EU-28 is actually absorbed in the EU-28, 1.6% in China, 2.6% in the USA, and 11.4% in RoW. In terms of value-added exports (lower panel), 44.1% of the EU-28 countries' value-added exports are absorbed in other EU member states, 5.6% in China, 9.2% in the USA, and 41.1% in RoW. Analogous interpretations hold for the other countries.

Table 3.5: Source and sink (in %)

Source \ Sink	Gross output					Value added				
	EU-28	China	USA	RoW	Total	EU-28	China	USA	RoW	Total
					Total					
EU-28	81.3	1.9	3.1	13.7	100.0	84.4	1.6	2.6	11.4	100.0
China	3.6	78.2	3.8	14.4	100.0	3.0	81.2	3.1	12.7	100.0
USA	2.2	0.8	89.8	7.2	100.0	2.0	0.7	91.1	6.3	100.0
RoW	4.9	4.1	5.0	86.1	100.0	4.2	3.2	4.2	88.4	100.0
Total	20.4	17.6	20.7	41.3	100.0	22.0	13.0	23.7	41.3	100.0
					Value added exports*					
EU-28	45.0	5.5	9.1	40.4	100.0	44.1	5.6	9.2	41.1	100.0
China	16.4	0.0	17.7	66.0	100.0	16.0	0.0	16.7	67.3	100.0
USA	21.4	8.2	0.0	70.4	100.0	22.2	7.8	0.0	70.0	100.0
RoW	19.4	16.1	19.5	45.0	100.0	19.6	15.1	19.4	45.9	100.0
Total	26.9	9.2	14.4	49.5	100.0	27.4	9.3	13.7	49.7	100.0

Note: *Including intra-regional trade for EU-28 and RoW.
Source: WIOD Release 2016; own calculations.

Structure of value-added content Table 3.6 allows for an interpretation in the value-added content assembled in a specific country for final use (either domestically or exported). Accordingly, EU-28 final goods assembly consists of 92.1% value added produced in the EU-28, 1.0% produced in China, 1.5% in the USA, and 5.5% in RoW (see upper panel of this table). When considering only final goods trade, the data (lower panel) tell us that the EU-28 final goods exports (incl. intra-EU trade flows), i.e. final goods assembled in an EU-28 member state and exported, consist of 85% of value added created in the EU-28, 1.9% in China, 3.1% in the USA, and 9.9% in RoW. Again, analogous interpretations hold for the other countries.

Table 3.6: Source and assembly (in %)

Source \ Assembly	Gross output					Value added				
	EU-28	China	USA	RoW	Total	EU-28	China	USA	RoW	Total
	Total final demand									
EU-28	90.2	1.2	1.9	4.1	21.3	92.1	1.5	1.4	3.7	23.0
China	1.7	90.8	1.5	3.6	19.7	1.0	89.4	0.7	2.2	13.8
USA	1.5	0.6	90.7	2.2	19.2	1.5	0.8	93.4	2.3	23.1
RoW	6.6	7.5	5.9	90.1	39.7	5.5	8.3	4.5	91.8	40.1
Total	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	Final goods exports*									
EU-28	84.7	1.4	2.9	5.5	32.9	85.0	2.1	2.5	6.0	35.6
China	2.9	87.4	2.7	5.8	22.2	1.9	84.1	1.5	4.2	16.4
USA	2.5	0.6	85.0	3.6	8.5	3.1	1.1	87.9	4.5	11.0
RoW	9.9	10.5	9.4	85.0	36.4	9.9	12.7	8.1	85.3	37.1
Total	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

Note: *Including intra-regional trade for EU-28 and RoW.
Source: WIOD Release 2016; own calculations.

4 Source-assembly-sink decompositions

In Section 2, the multi-country input-output table and some useful matrix notation have been introduced. Further, the gross output multiplier matrix (Leontief inverse) and the corresponding value-added multiplier matrix have been presented. In the previous Section 3 the 'source-sink' matrix (allowing for an interpretation in terms of value-added trade) **T** and the 'source-assembly' matrix **C** (allowing for an interpretation in terms of value-added content) have been presented. In this section, we provide some further decompositions of these characterisations of value-added flows in the global economy extending our matrix algebra. Specifically, based on this matrix algebra, we also reformulate the approach presented in Koopman et al. (2014) – referred to as KWW – which results in a bilateral representation of the KWW decomposition. This is presented in Appendix Section B. We have however to emphasise that the KWW approach is genuinely derived at the total economy level (i.e. not in a bilateral way) which will play a role when comparing the results. Consequently, some of the terms presented there are only comparable at the total economy level. However presenting it in a bilateral way allows one to study the differences to the approach outlined in this paper that will particularly be the content of Section 5, though selected similarities of this approach to KWW are already studied in this section.

4.1 Multiplier decomposition

The first step is to provide a decomposition of the multiplier matrices. To achieve this, we split the coefficients matrix, the gross output (Leontief inverse) and the value-added multiplier matrix into its diagonal and off-diagonal elements using the same notation as introduced in 2.[33] For example, the coefficients matrix is split into

$$\mathbf{A} = \begin{pmatrix} a^{11} & a^{12} & a^{13} \\ a^{21} & a^{22} & a^{23} \\ a^{31} & a^{32} & a^{33} \end{pmatrix} = \hat{\mathbf{A}} + \tilde{\mathbf{A}} = \begin{pmatrix} a^{11} & 0 & 0 \\ 0 & a^{22} & 0 \\ 0 & 0 & a^{33} \end{pmatrix} + \begin{pmatrix} 0 & a^{12} & a^{13} \\ a^{21} & 0 & a^{23} \\ a^{31} & a^{32} & 0 \end{pmatrix}$$

Matrices **L** and **B** are split analogously. Further, one can define the 'domestic' Leontief inverse by considering only the domestic parts (the diagonal elements) of the transactions matrix, i.e. $\hat{\mathbf{Z}}$ and the corresponding domestic parts (diagonal elements) of the coefficients matrix, i.e. $\hat{\mathbf{A}}$. The 'domestic' Leontief inverse is then calculated as $\bar{\mathbf{L}} = (\mathbf{I} - \hat{\mathbf{A}})^{-1}$, which is block-diagonal by definition. Note that in general, $\hat{\mathbf{L}} \neq \bar{\mathbf{L}}$, i.e. the diagonal elements of the global Leontief matrix are not equal to the diagonal elements of the domestic Leontief matrix. We define the difference as $\check{\mathbf{L}} = \hat{\mathbf{L}} - \bar{\mathbf{L}}$.[34] For further use, the global Leontief inverse is therefore split into the domestic Leontief inverse, the difference between the

[33] When including the industry dimension, this applies to the various blocks in the matrices.

[34] This difference has already been introduced and used in the analysis by Nagengast and Stehrer (2016) and recently applied in Arto et al. (2019). The elements of $\check{\mathbf{L}}$ are non-negative by definition.

domestic and the diagonal elements of the global Leontief, and the off-diagonal elements, thus[35]

$$\mathbf{L} = \begin{pmatrix} l^{11} & l^{12} & l^{13} \\ l^{21} & l^{22} & l^{23} \\ l^{31} & l^{32} & l^{33} \end{pmatrix} = \bar{\mathbf{L}} + \breve{\mathbf{L}} + \tilde{\mathbf{L}} = \begin{pmatrix} \bar{l}^{11} & 0 & 0 \\ 0 & \bar{l}^{22} & 0 \\ 0 & 0 & \bar{l}^{33} \end{pmatrix} + \begin{pmatrix} \breve{l}^{11} & 0 & 0 \\ 0 & \breve{l}^{22} & 0 \\ 0 & 0 & \breve{l}^{33} \end{pmatrix} + \begin{pmatrix} 0 & l^{12} & l^{13} \\ l^{21} & 0 & l^{23} \\ l^{31} & l^{32} & 0 \end{pmatrix}$$

Correspondingly the value-added multiplier matrix can be split into the domestic part, $\bar{\mathbf{B}} = \hat{\mathbf{v}}\bar{\mathbf{L}}$, the difference of this to the global Leontief elements $\breve{\mathbf{B}} = \hat{\mathbf{v}}\breve{\mathbf{L}} = \hat{\mathbf{v}}(\hat{\mathbf{L}} - \bar{\mathbf{L}})$[36] and the off-diagonal elements $\tilde{\mathbf{B}} = \hat{\mathbf{v}}\tilde{\mathbf{L}}$, thus resulting in

$$\mathbf{B} = \begin{pmatrix} b^{11} & b^{12} & b^{13} \\ b^{21} & b^{22} & b^{23} \\ b^{31} & b^{32} & b^{33} \end{pmatrix} = \bar{\mathbf{B}} + \breve{\mathbf{B}} + \tilde{\mathbf{B}} = \begin{pmatrix} \bar{b}^{11} & 0 & 0 \\ 0 & \bar{b}^{22} & 0 \\ 0 & 0 & \bar{b}^{33} \end{pmatrix} + \begin{pmatrix} \breve{b}^{11} & 0 & 0 \\ 0 & \breve{b}^{22} & 0 \\ 0 & 0 & \breve{b}^{33} \end{pmatrix} + \begin{pmatrix} 0 & b^{12} & b^{13} \\ b^{21} & 0 & b^{23} \\ b^{31} & b^{32} & 0 \end{pmatrix}$$

Though this might look like a purely definitional issue, it becomes crucial because it relates to and will explain the double-counting terms in the KWW approach (see Section 5 for details). The resulting values of these three parts of the multiplier matrices using the numerical example are provided in Table 4.1 (which therefore splits the numbers given in Table 2.3 and actually adds up to the respective totals). The

Table 4.1: Multiplier decomposition

	Gross output				Value added			
	EU-28	China	USA	RoW	EU-28	China	USA	RoW
Domestic multipliers with no border crossings								
EU-28	1.632	0.000	0.000	0.000	0.807	0.000	0.000	0.000
China	0.000	2.710	0.000	0.000	0.000	0.872	0.000	0.000
USA	0.000	0.000	1.669	0.000	0.000	0.000	0.921	0.000
RoW	0.000	0.000	0.000	1.807	0.000	0.000	0.000	0.816
Domestic multipliers with multiple border crossings								
EU-28	0.008	0.000	0.000	0.000	0.003	0.000	0.000	0.000
China	0.000	0.019	0.000	0.000	0.000	0.005	0.000	0.000
USA	0.000	0.000	0.009	0.000	0.000	0.000	0.004	0.000
RoW	0.000	0.000	0.000	0.017	0.000	0.000	0.000	0.007
International multipliers								
EU-28	0.233	0.038	0.037	0.098	0.095	0.016	0.016	0.041
China	0.038	0.000	0.029	0.087	0.011	0.000	0.008	0.025
USA	0.034	0.017	0.000	0.050	0.018	0.008	0.000	0.025
RoW	0.156	0.258	0.123	0.206	0.066	0.099	0.052	0.087
Total	2.102	3.042	1.867	2.265	1.000	1.000	1.000	1.000

Note 1): Domestic multipliers include intra-country flows for EU-28 and RoW.
Note 2): International multipliers include inter-country multipliers for EU-28 and RoW.
Source: WIOD Release 2016; own calculations.

[35] For technical details and how this links to the power expansion of the Leontief inverse, see Appendix Section A. One can also interpret this as a special case of the 'hypothetical extraction method' where all off-diagonal elements are block-wise set to **0** (see Section 6 and Appendix Section A for a details).

[36] By definition, it holds that $\bar{\mathbf{B}} + \breve{\mathbf{B}} = \hat{\mathbf{B}}$.

block-diagonal elements are split into 'pure' domestic linkages[37] and multipliers including multiple border crossings $\breve{\mathbf{L}}$ (for details, see Appendix Section A, where this becomes clear when developing the Leontief inverse as a power expansion). The off-diagonal elements are just split out of the multiplier matrices, however EU-28 and RoW include inter-country multiplier effects within these groups (therefore, for these countries there are Also entries in the diagonal cells in the lower panel of Table 4.1. These decompositions of the multiplier matrices are now used to decompose the 'source-sink' and the 'source-assembly' matrices introduced in the previous section.

4.2 Decomposition of the 'source-sink' matrix

4.2.1 Domestic consumption and exports of value added

Based on this decomposition of the multiplier matrix and splitting the final demand matrix into the diagonal and the off-diagonal elements, the 'source-sink' matrix can be split into seven components, as shown in equation (4.1). For reasons outlined below the matrix $\tilde{\mathbf{B}}\tilde{\mathbf{F}}$ is again split into its diagonal and off-diagonal blocks, i.e. $\tilde{\mathbf{B}}\tilde{\mathbf{F}} = \widehat{\tilde{\mathbf{B}}\tilde{\mathbf{F}}} + \widetilde{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}$.

$$\mathbf{T} = \mathbf{BF} = \underbrace{\overbrace{(\bar{\mathbf{B}}\hat{\mathbf{F}} + \breve{\mathbf{B}}\hat{\mathbf{F}})}^{=\hat{\mathbf{B}}\hat{\mathbf{F}}} + \widehat{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}}_{\text{Domestic consumption}} + \underbrace{\overbrace{(\bar{\mathbf{B}}\tilde{\mathbf{F}} + \breve{\mathbf{B}}\tilde{\mathbf{F}})}^{=\hat{\mathbf{B}}\tilde{\mathbf{F}}} + \tilde{\mathbf{B}}\hat{\mathbf{F}} + \widetilde{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}}_{\text{Value added exports}} \tag{4.1}$$

The terms are arranged in a way that the first three terms comprise domestic absorption of value added, whereas the remaining four terms add up to the countries' bilateral value-added exports (i.e.Â absorption of domestically produced value added abroad). We refer to these seven terms as VAT1 to VAT7 and discuss them individually. For a neat interpretation, it proves insightful to look at the details considering three countries.

Domestic consumption: The first element $\hat{\mathbf{B}}\hat{\mathbf{F}}$ (VAT1+VAT2) comprises the value-added flows with the generation of value added (source), the final assembly stage, and final absorption (sink) taking place in the same country. These flows are split into the two components, i.e. $\hat{\mathbf{B}}\hat{\mathbf{F}} = (\bar{\mathbf{B}}\hat{\mathbf{F}} + \breve{\mathbf{B}}\hat{\mathbf{F}})$, resulting in

$$\hat{\mathbf{B}}\hat{\mathbf{F}} = \begin{pmatrix} b^{11}f^{11} & 0 & 0 \\ 0 & b^{22}f^{22} & 0 \\ 0 & 0 & b^{33}f^{33} \end{pmatrix} = \begin{pmatrix} \bar{b}^{11}f^{11} & 0 & 0 \\ 0 & \bar{b}^{22}f^{22} & 0 \\ 0 & 0 & \bar{b}^{33}f^{33} \end{pmatrix} + \begin{pmatrix} \breve{b}^{11}f^{11} & 0 & 0 \\ 0 & \breve{b}^{22}f^{22} & 0 \\ 0 & 0 & \breve{b}^{33}f^{33} \end{pmatrix}$$

The first term is value added generated in the source economy which – because this part of the Leontief inverse element includes domestic linkages captured in $\bar{\mathbf{B}}$ only – never leaves the country. Therefore this con-

[37]The diagonal entries in the first panel in Table 4.1 for EU-28 and RoW indicate the multipliers aggregated over the countries in the respective group.

stitutes the purely domestic part of the value chain and is characterised as $[\textit{source}^r \circlearrowleft^r \textit{assembly}^r \rightarrow \textit{sink}^r]$. The second term is value added generated in a source country that leaves the country (embodied in intermediate products). After various production stages across all countries in the world (eventually including the country of origin), the value added is ultimately assembled as part of a final product in the source country and also absorbed there. Therefore it can be characterised as $[\textit{source}^r \rightsquigarrow^{\forall c} \textit{assembly}^r \rightarrow \textit{sink}^r]$.

The third term (VAT3) consists of the diagonal blocks of matrix $\tilde{\mathbf{B}}\tilde{\mathbf{F}}$. The diagonal and off-diagonal elements will have a different interpretation and are therefore split according to $\tilde{\mathbf{B}}\tilde{\mathbf{F}} = \widehat{\tilde{\mathbf{B}}\tilde{\mathbf{F}}} + \widetilde{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}$. This matrix takes the form

$$\tilde{\mathbf{B}}\tilde{\mathbf{F}} = \begin{pmatrix} b^{12}f^{21} + b^{13}f^{31} & 0 & 0 \\ 0 & b^{21}f^{21} + b^{23}f^{32} & 0 \\ 0 & 0 & b^{31}f^{13} + b^{32}f^{23} \end{pmatrix} + \begin{pmatrix} 0 & b^{13}f^{32} & b^{12}f^{23} \\ b^{23}f^{31} & 0 & b^{21}f^{13} \\ b^{32}f^{21} & b^{31}f^{12} & 0 \end{pmatrix}$$

A typical cell of matrix $\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}$ denotes the value added generated in a source country, which after many border crossings is assembled into a final product in another country. This final product is then shipped back to the source country. Because this is value added that flows back to the country of origin embodied in a final product imported from another country, these terms constitute 're-imports of value added' and are included as domestic consumption in equation (4.1). These flows are accordingly characterised as $[\textit{source}^r \rightsquigarrow^{\forall c} \textit{assembly}^s \rightarrow \textit{sink}^r]$.

Value-added exports: The fourth and fifth term in equation (4.1), i.e. VAT4 and VAT5, comprise value-added exports of products finally assembled in the source country.

$$\hat{\mathbf{B}}\tilde{\mathbf{F}} = \begin{pmatrix} 0 & b^{11}f^{12} & b^{11}f^{13} \\ b^{22}f^{21} & 0 & b^{22}f^{23} \\ b^{33}f^{31} & b^{33}f^{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \bar{b}^{11}f^{12} & \bar{b}^{11}f^{13} \\ \bar{b}^{22}f^{21} & 0 & \bar{b}^{22}f^{23} \\ \bar{b}^{33}f^{31} & \bar{b}^{33}f^{32} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \breve{b}^{11}f^{12} & \breve{b}^{11}f^{13} \\ \breve{b}^{22}f^{21} & 0 & \breve{b}^{22}f^{23} \\ \breve{b}^{33}f^{31} & \breve{b}^{33}f^{32} & 0 \end{pmatrix}$$

Analogous to the above, the first matrix $\bar{\mathbf{B}}\tilde{\mathbf{F}}$ includes value added leaving the source country only as part of the final product, which is absorbed in the sink country. This is therefore characterised as $[\textit{source}^r \circlearrowleft^r \textit{assembly}^r \rightarrow \textit{sink}^s]$. And, as well analogous to above, the second matrix $\breve{\mathbf{B}}\tilde{\mathbf{F}}$ can be interpreted as $[\textit{source}^r \rightsquigarrow^{\forall c} \textit{assembly}^r \rightarrow \textit{sink}^s]$ accordingly.

The sixth element (VAT6) indicate the value added generated in a source country, embodied in intermediate products that are finally assembled and absorbed in the sink country and is given by

$$\tilde{\mathbf{B}}\hat{\mathbf{F}} = \begin{pmatrix} 0 & b^{12}f^{22} & b^{13}f^{33} \\ b^{21}f^{11} & 0 & b^{23}f^{33} \\ b^{31}f^{11} & b^{32}f^{22} & 0 \end{pmatrix}$$

These flows can therefore be characterised as *[source^r $\rightsquigarrow^{\forall c}$ assembly^s $\rightarrow$ sink^s]*.

The final, seventh, term in equation (4.1) is the off-diagonal elements of the matrix $\tilde{\mathbf{B}}\tilde{\mathbf{F}}$ discussed above. These indicate the value added generated in the source country, which after many production stages are finally assembled in a another country, and then exported to and absorbed in a third country (sink). These value-added flows can therefore be characterised as *[source^r $\rightsquigarrow^{\forall c}$ assembly^s $\rightarrow$ sink^t]*.

4.2.2 Numerical example and relation to literature

Table 4.2 provides the numbers for the numerical example. Using this example, we also indicate how the results from this approach compare to the results from the KWW decomposition (see Table B.1 in the Appendix). A further more technical discussion is provided in Section 5). The panels are ordered according to the terms in equation (4.1), i.e. the first three panels correspond to domestic absorption of value added, whereas the remaining ones correspond to value-added exports.

Domestic consumption: The first and second panels in Table 4.2 report the terms VAT1 and VAT2. The (pure) intra-country flows, VAT1, do not appear in the KWW decomposition, which does not include domestic absorption. Interestingly, the second term, VAT2, corresponds to the 'domestic value added in exports re-imported as intermediary inputs' (KWW5) when compared to the entries in Table B.1 in the Appendix. This is consistent with our interpretation provided above as *[source^r $\rightsquigarrow^{\forall c}$ assembly^r $\rightarrow$ sink^r]*. Technically, this implies that $\breve{\mathbf{B}}\hat{\mathbf{F}} = \widehat{\tilde{\mathbf{B}}\tilde{\mathbf{A}}\bar{\mathbf{L}}\mathbf{F}}$ (see proof below). The third term of domestic consumption, VAT3, equals 'domestic value added re-imported as final goods' (KWW4).

Proof that VAT2=KWW5: This can be shown analytically by using that $\breve{\mathbf{B}}\hat{\mathbf{F}} = \hat{\mathbf{B}}\hat{\mathbf{F}} - \bar{\mathbf{B}}\hat{\mathbf{F}}$, which when expressed in the form of the (diagonalised) value-added coefficients vector and the Leontief inverse becomes $\breve{\mathbf{B}}\hat{\mathbf{F}} = \hat{\mathbf{v}}\hat{\mathbf{L}}\hat{\mathbf{F}} - \hat{\mathbf{v}}\bar{\mathbf{L}}\hat{\mathbf{F}}$. Using the property of inverse matrices (see Appendix Section C), the block-diagonal elements can be written as $\hat{\mathbf{L}} = \widehat{\tilde{\mathbf{L}}\tilde{\mathbf{A}}\bar{\mathbf{L}}} + \bar{\mathbf{L}} = \widehat{\tilde{\mathbf{L}}\tilde{\mathbf{A}}\bar{\mathbf{L}}} + \bar{\mathbf{L}}$. Inserting this expression into the previous equation shows that these two expressions are equivalent, $\hat{\mathbf{v}}[\widehat{\tilde{\mathbf{L}}\tilde{\mathbf{A}}\bar{\mathbf{L}}} + \bar{\mathbf{L}}]\hat{\mathbf{F}} - \hat{\mathbf{v}}\bar{\mathbf{L}}\hat{\mathbf{F}} = \hat{\mathbf{v}}\widehat{\tilde{\mathbf{L}}\tilde{\mathbf{A}}\bar{\mathbf{L}}}\hat{\mathbf{F}} + \hat{\mathbf{v}}\bar{\mathbf{L}}\hat{\mathbf{F}} - \hat{\mathbf{v}}\bar{\mathbf{L}}\hat{\mathbf{F}} = \widehat{\tilde{\mathbf{B}}\tilde{\mathbf{A}}\bar{\mathbf{L}}\mathbf{F}}$. □

Value-added exports: The four panels, VAT4 to VAT7, show the components of value-added exports in this approach.[38] VAT4 and VAT5 sum up to the 'domestic value added in direct final goods exports' (KWW1) (compare to Appendix Table B.1). Here, these value-added exports are decomposed into the pure domestic part and the one which involves multiple border crossings. The second term is crucial when explaining the double counting terms in the gross exports decomposition provided in KWW (discussed

[38]Note that in this table, the diagonal cells for the EU-28 and RoW are not equal to zero. The reason is the same as that above. For each of the individual countries in these groups, these are zero, when aggregating over countries (for sake of exposition), these include flows across the countries in the respective country groups.

Table 4.2: Decomposition of the value-added trade matrix

	Gross output					Value added				
	EU-28	China	USA	RoW	Total	EU-28	China	USA	RoW	Total
					Domestic consumption					
VAT1: *[source^r $\circlearrowleft^r$ assembly^r $\rightarrow$ sink^r]*										
	$\bar{\mathbf{L}}\hat{\mathbf{F}}$					$\bar{\mathbf{B}}\hat{\mathbf{F}}$				
EU-28	22,509	0	0	0	22,509	12,445	0	0	0	12,445
China	0	24,617	0	0	24,617	0	8,382	0	0	8,382
USA	0	0	27,548	0	27,548	0	0	15,737	0	15,737
RoW	0	0	0	47,018	47,018	0	0	0	23,511	23,511
Total	22,509	24,617	27,548	47,018	121,692	12,445	8,382	15,737	23,511	60,075
VAT2 (=KWW5): *[source^r $\rightsquigarrow^{\forall c}$ assembly^r $\rightarrow$ sink^r]*										
	$\breve{\mathbf{L}}\hat{\mathbf{F}}$					$\breve{\mathbf{B}}\hat{\mathbf{F}}$				
EU-28	72	0	0	0	72	30	0	0	0	30
China	0	137	0	0	137	0	38	0	0	38
USA	0	0	128	0	128	0	0	61	0	61
RoW	0	0	0	355	355	0	0	0	140	140
Total	72	137	128	355	693	30	38	61	140	270
VAT3 (=KWW4): *[source^r $\rightsquigarrow^{\forall c}$ assembly^s $\rightarrow$ source^r]*										$\hat{A}$
	$\widehat{\tilde{\mathbf{L}}\tilde{\mathbf{F}}}$					$\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}$				
EU-28	99	0	0	0	99	41	0	0	0	41
China	0	76	0	0	76	0	21	0	0	21
USA	0	0	134	0	134	0	0	61	0	61
RoW	0	0	0	336	336	0	0	0	125	125
Total	99	76	134	336	644	41	21	61	125	248
					Value added exports					
VAT4 (= part of KWW1): *[source^r $\circlearrowleft^r$ assembly^r $\rightarrow$ sink^s]*										
	$\bar{\mathbf{L}}\tilde{\mathbf{F}}$					$\bar{\mathbf{B}}\tilde{\mathbf{F}}$				
EU-28	2,313	229	344	1,816	4,701	924	97	144	765	1,930
China	550	0	663	2,392	3,605	149	0	177	682	1,008
USA	228	85	0	883	1,197	113	38	0	430	581
RoW	996	552	1,090	2,309	4,947	380	201	408	905	1,894
Total	4,087	866	2,096	7,401	14,450	1,566	335	729	2,782	5,412
VAT5 (= part of KWW1): *[source^r $\hat{A}$ $\rightsquigarrow^{\forall c}$ assembly^r $\rightarrow$ sink^s]*										
	$\breve{\mathbf{L}}\tilde{\mathbf{F}}$					$\breve{\mathbf{B}}\tilde{\mathbf{F}}$				
EU-28	21	3	4	15	43	8	1	2	6	17
China	7	0	9	29	45	2	0	2	8	12
USA	2	1	0	7	10	1	0	0	3	5
RoW	13	9	12	21	54	5	3	5	8	20
Total	43	13	25	73	153	16	5	9	25	55
VAT6 (=KWW2): *[source^r $\rightsquigarrow^{\forall c}$ assembly^s $\rightarrow$ sink^s]*										
	$\tilde{\mathbf{L}}\hat{\mathbf{F}}$					$\tilde{\mathbf{B}}\hat{\mathbf{F}}$				
EU-28	2,298	319	549	2,261	5,427	952	133	231	957	2,272
China	392	0	439	1,878	2,708	110	0	116	557	783
USA	328	147	0	1,169	1,644	170	71	0	573	814
RoW	1,547	1,899	1,748	4,312	9,507	660	723	732	1,810	3,926
Total	4,565	2,365	2,736	9,620	19,285	1,893	927	1,079	3,896	7,795
VAT7 (=KWW3): *[source^r $\rightsquigarrow^{\forall c}$ assembly^s $\rightarrow$ sink^t]*										
	$\widetilde{\tilde{\mathbf{L}}\tilde{\mathbf{F}}}$					$\widetilde{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}$				
EU-28	621	96	166	626	1,509	248	39	68	256	610
China	183	0	111	263	557	51	0	31	72	154
USA	119	25	0	166	310	61	13	0	83	157
RoW	584	143	315	661	1,703	233	56	119	263	671
Total	1,507	265	592	1,715	4,079	593	107	217	674	1,592

Note: Values in bn USD.
Source: WIOD Release 2016; own calculations.

in detail in Section 5). The next panel corresponds to the 'intermediary exports absorbed by partner', thus VAT6=KWW2. The bottom panel reports the values of the 'intermediary exports re-exported' (VAT7=KWW3), which also appears in the KWW decomposition.Thus, this source-sink decomposition approach leads to the same results as the KWW approach in a bilateral perspective and additionally splits the term KWW1 into two components.

4.3 Decomposing the 'source-assembly' matrix

Using the same method, the 'source-assembly' matrix $\mathbf{C}$ can be decomposed similarly. Some of the terms appearing are - by definition - equal to those in the decomposition of the 'source-sink' matrix, whereas some new terms appear. Depending on the exact matrix manipulations applied, two decompositions are possible concerning final goods exports: (i) the bilateral factor contents of a country's total final goods exports, and (ii) the total factor contents of a country's bilateral final goods exports. The latter will be discussed separately in the next subsection.

4.3.1 Domestic and foreign content of domestic absorption and total final goods exports

Using the matrix manipulations explained above $\mathbf{C}$ is split into six terms:

$$\mathbf{C} = \mathbf{B}(\widehat{\mathbf{F1}}) = \mathbf{B}\hat{\mathbf{f}} = \underbrace{\overbrace{\bar{\mathbf{B}}(\widehat{\hat{\mathbf{F}}\mathbf{1}}) + \breve{\mathbf{B}}(\widehat{\hat{\mathbf{F}}\mathbf{1}})}^{\hat{\mathbf{B}}(\widehat{\hat{\mathbf{F}}\mathbf{1}})} + \tilde{\mathbf{B}}(\widehat{\hat{\mathbf{F}}\mathbf{1}})}_{\text{Domestic consumption}} + \underbrace{\overbrace{\bar{\mathbf{B}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}}) + \breve{\mathbf{B}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}})}^{\hat{\mathbf{B}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}})} + \tilde{\mathbf{B}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}})}_{\text{Final goods exports}} \tag{4.2}$$

that are grouped together according to domestic consumption versus final goods exports.[39] The terms are referred to as VAC1 to VAC6.

Domestic consumption: The first two terms in equation (4.2) are identical to $\bar{\mathbf{B}}\hat{\mathbf{F}}$ (VAT1) and $\breve{\mathbf{B}}\hat{\mathbf{F}}$ (VAT2), as already discussed in the previous section. The third term (VAC3) in equation (4.2) includes the off-diagonal elements of the domestic consumption matrix and comprise the 'domestic value added in intermediate goods exports absorbed by direct importers', i.e. $\tilde{\mathbf{B}}\hat{\mathbf{F}}$ or VAT6(=KWW2). In this context, it can also be interpreted as the foreign value-added content of the sink country's final goods consumption (or, interpreted differently, the imports of value added).

[39] Alternatively, one could group together according to domestic versus foreign content

$$\mathbf{C} = \mathbf{B}(\widehat{\mathbf{F1}}) = \mathbf{B}\hat{\mathbf{f}} = \underbrace{\overbrace{\bar{\mathbf{B}}(\widehat{\hat{\mathbf{F}}\mathbf{1}}) + \breve{\mathbf{B}}(\widehat{\hat{\mathbf{F}}\mathbf{1}})}^{\hat{\mathbf{B}}(\widehat{\hat{\mathbf{F}}\mathbf{1}})} + \overbrace{\bar{\mathbf{B}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}}) + \breve{\mathbf{B}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}})}^{\hat{\mathbf{B}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}})}}_{\text{Domestic content}} + \underbrace{\tilde{\mathbf{B}}(\widehat{\hat{\mathbf{F}}\mathbf{1}}) + \tilde{\mathbf{B}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}})}_{\text{Foreign content}}$$

The domestic content is the own value added absorbed in the country of assembly or value added exported in the form of final products. The foreign content is the foreign value added absorbed in one country with products finally assembled in the same country, or of the products finally assembled in this country and exported (in form of final products. The foreign content will be discussed in an alternative way later.

Final goods exports: The second part of equation (4.2) shows the domestic and foreign contents of a country's total final goods exports (i.e. summed over trading partners). In this decomposition, the domestic content of a source country's total exports of final goods $\hat{\mathbf{B}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}})$ is broken down into the purely domestic value added (VAC4) and the value added that crosses borders several times with the product finally assembled in this country and further shipped as a final product (VAC5), i.e.

$$\hat{\mathbf{B}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}}) = \begin{pmatrix} \bar{b}^{11}(f^{12}+f^{13}) & 0 & 0 \\ 0 & \bar{b}^{22}(f^{21}+f^{23}) & 0 \\ 0 & 0 & \bar{b}^{33}(f^{31}+f^{32}) \end{pmatrix} + \begin{pmatrix} \breve{b}^{11}(f^{12}+f^{13}) & 0 & 0 \\ 0 & \breve{b}^{22}(f^{21}+f^{23}) & 0 \\ 0 & 0 & \breve{b}^{33}(f^{31}+f^{32}) \end{pmatrix}$$

By definition, the row sums of these matrices are equal to the row sums of VAT4 and VAT5 (which together sum up to KWW1) as $\bar{\mathbf{B}}\tilde{\mathbf{F}}\mathbf{1} = \bar{\mathbf{B}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}})\mathbf{1}$ and $\breve{\mathbf{B}}\tilde{\mathbf{F}}\mathbf{1} = \breve{\mathbf{B}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}})\mathbf{1}$. The reason is that in the previous section, these terms denote the value-added exports of a country's bilateral final goods exports, where here the terms denote a country's domestic value-added content of total final goods exports. Finally, the last term VAC6 in equation (4.2) shows the flows

$$\tilde{\mathbf{B}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}}) = \begin{pmatrix} 0 & b^{12}(f^{21}+f^{23}) & b^{13}(f^{31}+f^{32}) \\ b^{21}(f^{12}+f^{13}) & 0 & b^{23}(f^{31}+f^{32}) \\ b^{31}(f^{12}+f^{13}) & b^{32}(f^{21}+f^{23}) & 0 \end{pmatrix}$$

i.e. the bilateral foreign content of an assembly country's total final goods exports. The off-diagonal cells of this matrix represent flows where the country of final assembly differs from the source country. The final products areÂ shipped from the country of assembly to third countries, which can either be the original source country of value added or a another third country. For the former case, this would constitute the 'domestic value added in intermediate goods exports re-imported as final products'), i.e. *[source*^{r} $\rightsquigarrow^{\forall c}$ *assembly*^{s} $\rightarrow$ *sink*^{r}*]*, whereas, for the latter case, it is the 'domestic value added in intermediate goods exports re-exported to third countries', i.e. *[source*^{r} $\rightsquigarrow^{\forall c}$ *assembly*^{s} $\rightarrow$ *sink*^{t}*]*. The former interpretation has already appeared as the diagonal matrix $\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}, i.e.$VAT3(=KWW4)), whereas the latter has appeared as $\widetilde{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}$, i.e. VAT7=(KWW3). Clearly, this implies that the row sums in both cases have to be the same. Mathematically, this is clear because $(\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{F}}} + \widetilde{\tilde{\mathbf{B}}\tilde{\mathbf{F}}})\mathbf{1} = \tilde{\mathbf{B}}\tilde{\mathbf{F}}\mathbf{1} = \tilde{\mathbf{B}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}})\mathbf{1}$. Though the matrices differ given the different interpretations of the sink-source versus sink-assembly matrices, they characterise the same value-added flows: The domestic value added re-imported in final goods and the domestic value added re-exported to third countries (embodied in final goods) in the sink-source interpretation equals the foreign value-added content of a country's final goods exports.

4.3.2 Numerical example

The numerical results for the domestic part are already reported in Table 4.2. Therefore, in Table 4.3, only the terms for final goods exports are shown. By definition, the off-diagonal elements of the first two

Table 4.3: Decomposition of the value-added content of (total) final goods exports

	Gross output					Value added				
	EU-28	China	USA	RoW	Total	EU-28	China	USA	RoW	Total
VAC4[=RowSum VAT4(=KWW1)]: *[source^r $\circlearrowleft^r$ assembly^r $\rightarrow$ sink^s]*										
	$\bar{\mathbf{L}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}})$					$\bar{\mathbf{B}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}})$				
EU-28	4,701	0	0	0	4,701	1,930	0	0	0	1,930
China	0	3,605	0	0	3,605	0	1,008	0	0	1,008
USA	0	0	1,197	0	1,197	0	0	581	0	581
RoW	0	0	0	4,947	4,947	0	0	0	1,894	1,894
Total	4,701	3,605	1,197	4,947	14,450	1,930	1,008	581	1,894	5,412
VAC5[=RowSum VAT5(=KWW1)]: *[source^r $\rightsquigarrow^{\forall c}$ assembly^r $\rightarrow$ sink^s]*										
	$\breve{\mathbf{L}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}})$					$\breve{\mathbf{B}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}})$				
EU-28	43	0	0	0	43	17	0	0	0	17
China	0	45	0	0	45	0	12	0	0	12
USA	0	0	10	0	10	0	0	5	0	5
RoW	0	0	0	54	54	0	0	0	20	20
Total	43	45	10	54	153	17	12	5	20	55
VAC6[= Column sums equal to row sum of KWW7]: *[source^r $\rightsquigarrow^{\forall c}$ assembly^s $\rightarrow$ sink^t]*										
	$\tilde{\mathbf{L}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}})$					$\tilde{\mathbf{B}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}})$				
EU-28	1,133	60	41	374	1,608	453	25	17	156	651
China	201	0	38	394	633	55	0	10	110	175
USA	170	27	0	247	444	88	13	0	117	219
RoW	690	438	134	777	2,039	279	155	54	307	795
Total	2,193	525	213	1,793	4,724	875	193	81	691	1,840

Note: Values in bn USD.
Source: WIOD Release 2016; own calculations.

panels are zero. Comparison with Table 4.2 shows that the row sums of VAT4 and VAT5 coincide with the row sums of VAC4 and VAC5, as these constitute the 'domestic value added in direct final goods exports' (KWW1), but are here shown in a country's total exports, as argued above. Considering the last panel in this table, one can verify that the row sums are equal to the row sum of VAT3 plus VAT7 (also already discussed above). For later records, it is also noted that the column sums of this matrix are equal to the row sums of KWW7 (see Appendix Table B.1) because it is the total foreign content of a country's final goods exports, which is discussed in more detail in the next subsection.

4.4 Decomposing the 'assembly-sink' matrix

As already mentioned above (see Section 3), the final demand matrix can be considered an 'assembly-sink' matrix because a typical element f^{rc} indicates the country of final assembly r and the sink-country c. Further, the two decompositions presented in the previous subsections did not provide a decomposition of the bilateral final goods exports (but only the bilateral factor contents of a country's total final goods exports). By contrast, here we now consider the total factor contents (split into domestic and total foreign content) of a country's bilateral final goods exports. This is achieved by rearranging the value-added multiplier matrix as $\hat{\mathbf{B}} + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}}) = \mathbf{I}$ or, in detail,

$$\hat{\mathbf{B}} + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}}) = \begin{pmatrix} b^{11} & 0 & 0 \\ 0 & b^{22} & 0 \\ 0 & 0 & b^{33} \end{pmatrix} + \begin{pmatrix} (b^{21}+b^{31}) & 0 & 0 \\ 0 & (b^{12}+b^{32}) & 0 \\ 0 & 0 & (b^{13}+b^{23}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which by definition sums up to the identity matrix. This can be used to split the assembly-sink (final demand) matrix into the following six terms:

$$\mathbf{F} = \underbrace{(\bar{\mathbf{B}}\hat{\mathbf{F}} + \breve{\mathbf{B}}\hat{\mathbf{F}}) + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\hat{\mathbf{F}}}_{\text{Domestic consumption}} + \underbrace{(\bar{\mathbf{B}}\tilde{\mathbf{F}} + \breve{\mathbf{B}}\tilde{\mathbf{F}}) + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{F}}}_{\text{Final goods exports}} \tag{4.3}$$

The first, second, fourth and fifth terms have already appeared in the previous decompositions and, in this context, are the domestic content (pure or with multiple border crossings) of domestically assembled products that are absorbed domestically or exported. The third term

$$(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\hat{\mathbf{F}} = \begin{pmatrix} (b^{21}+b^{31})f^{11} & 0 & 0 \\ 0 & (b^{12}+b^{32})f^{22} & 0 \\ 0 & 0 & (b^{13}+b^{23})f^{33} \end{pmatrix}$$

denotes the foreign value-added content of domestically assembled and domestically absorbed final goods. The column sums of this matrix correspond to the column sums of KWW2 (Domestic value added in intermediate goods exports absorbed by direct importers). Finally, the sixth term reads

$$(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{F}} = \begin{pmatrix} 0 & (b^{21}+b^{31})f^{12} & (b^{21}+b^{31})f^{13} \\ (b^{12}+b^{32})f^{21} & 0 & (b^{12}+b^{32})f^{23} \\ (b^{13}+b^{23})f^{31} & (b^{13}+b^{23})f^{32} & 0 \end{pmatrix}$$

and includes the (total) foreign content of bilateral final goods exports. This is the bilateral pendant to the term KWW7 (see Appendix B and the numbers reported in Appendix Table B.1). The corresponding numbers for the third and sixth terms are reported in Table 4.4. Technically, in the previous section, the term $\tilde{\mathbf{B}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}})$ has included the (i) bilateral foreign content of an assembly country's total final goods exports, whereas this matrix includes (ii) the total foreign content of an assembly country's bilateral final goods exports. This also explains why the column sums of the former matrix are equal to the row sums of this matrix.[40]

[40]Formally this follows from $[\mathbf{1}'\tilde{\mathbf{B}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}})]' = [\mathbf{1}'(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})(\widehat{\tilde{\mathbf{F}}\mathbf{1}})]' = [\mathbf{1}'(\widehat{\tilde{\mathbf{F}}\mathbf{1}})(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})]' = (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})(\widehat{\tilde{\mathbf{F}}\mathbf{1}})\mathbf{1} = (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{F}}\mathbf{1}$.

Table 4.4: Decomposition of the assembly-sink matrix

	Gross output					Value added				
	EU-28	China	USA	RoW	Total	EU-28	China	USA	RoW	Total
Term 3										
	$\widehat{(\mathbf{1}'\tilde{\mathbf{L}})}\hat{\mathbf{F}}$					$\widehat{(\mathbf{1}'\tilde{\mathbf{B}})}\hat{\mathbf{F}}$				
EU-28	4,565	0	0	0	4,565	1,893	0	0	0	1,893
China	0	2,365	0	0	2,365	0	927	0	0	927
USA	0	0	2,736	0	2,736	0	0	1,079	0	1,079
RoW	0	0	0	9,620	9,620	0	0	0	3,896	3,896
Total	4,565	2,365	2,736	9,620	19,285	1,893	927	1,079	3,896	7,795
Term 6										
	$\widehat{(\mathbf{1}'\tilde{\mathbf{L}})}\tilde{\mathbf{F}}$					$\widehat{(\mathbf{1}'\tilde{\mathbf{B}})}\tilde{\mathbf{F}}$				
EU-28	1,142	104	151	796	2,193	454	40	60	321	875
China	82	0	102	341	525	30	0	37	126	193
USA	41	20	0	152	213	15	7	0	58	81
RoW	341	217	473	763	1,793	135	81	181	294	691
Total	1,606	341	726	2,051	4,724	634	128	279	799	1,840

Note: Values in bn USD.
Source: WIOD Release 2016; own calculations.

4.5 Summary

In this section, we provided the decomposition of the source-sink, the source-assembly and the assembly-sink (final demand) matrices and provided interpretations of the resulting terms. Table 4.5 gives an overview of what has been achieved so far and how the decompositions presented are related to each other, and also in relation to the KWW approach. Note that the approach in this paper focuses on and

Table 4.5: Comparison

Source-sink		Source-assembly		Assembly-sink		KWW
$\mathbf{T} = \mathbf{BF}$		$\mathbf{C} = \mathbf{B}(\widehat{\mathbf{F1}})$		$\mathbf{F}$		
Domestic absorption						
$\bar{\mathbf{B}}\hat{\mathbf{F}}$	=	$\bar{\mathbf{B}}(\widehat{\hat{\mathbf{F}}\mathbf{1}})$	=	$\bar{\mathbf{B}}\hat{\mathbf{F}}$		
$\breve{\mathbf{B}}\hat{\mathbf{F}}$	=	$\breve{\mathbf{B}}(\widehat{\hat{\mathbf{F}}\mathbf{1}})$	=	$\breve{\mathbf{B}}\hat{\mathbf{F}}$	=	KWW5[1)]
$\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}$	=				=	KWW4
				$\widehat{(\mathbf{1}'\tilde{\mathbf{B}})}\hat{\mathbf{F}}\hat{\mathrm{A}}$	${}^{C}{=}^{C}$	KWW2
Foreign absorption						
$\bar{\mathbf{B}}\tilde{\mathbf{F}}$	${}^{R}{=}^{R}$	$\bar{\mathbf{B}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}})$	${}^{R}{=}^{R}$	$\bar{\mathbf{B}}\tilde{\mathbf{F}}$	=	KWW1(1)
$\breve{\mathbf{B}}\tilde{\mathbf{F}}$	${}^{R}{=}^{R}$	$\breve{\mathbf{B}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}})$	${}^{R}{=}^{R}$	$\breve{\mathbf{B}}\tilde{\mathbf{F}}$	=	KWW1(2)
$\tilde{\mathbf{B}}\hat{\mathbf{F}}$	=	$\tilde{\mathbf{B}}(\widehat{\hat{\mathbf{F}}\mathbf{1}})$			=	KWW2
$\widetilde{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}$	=				=	KWW3
		$\tilde{\mathbf{B}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}})$	${}^{C}{=}^{R}$			
			${}^{C}{=}^{R}$	$\widehat{(\mathbf{1}'\tilde{\mathbf{B}})}\tilde{\mathbf{F}}$	=	KWW7

KWW1: DVA in direct final goods exports; KWW2: DVA in intermediate goods exports absorbed by direct importers
KWW3: DVA in intermediate goods exports re-exported to third countries; KWW4: DVA in intermediate goods exports re-imported as final goods; KWW5: DVA in intermediate goods exports re-imported as intermediate goods and finally absorbed at home (see proof in Section 4.2.2); KWW7: FVA in exports of final goods

results in bilateral value-added flows, whereas KWW explored total exports. Nonetheless, in a number of cases, the KWW also yields correspondences to the bilateral flows, as indicated by the equal signs. In some cases, only the row or column sums coincide, which is indicated by 'R' or 'C' in the table. As one can see, the three decompositions presented in this section reproduce KWW1 to KWW5 and KWW7 in various contexts. In addition, the term capturing the 'domestic value added in direct final goods exports' has been split into two. This will become important for understanding the gross export decomposition presented in the next section. However, various terms appearing in the KWW-decomposition are missing: these are the double counting terms, KWW6 and KWW9, and KWW8 ('foreign value added in the exports of intermediate goods'). This is clear because so far, the approach focused on final demand (domestic or traded), whereas KWW has been motivated by a decomposition of gross export flows, including intermediate goods trade. Such a decomposition of gross export flows using the method introduced here is presented in the next section.

5 Decomposition of bilateral gross export flows

In this section, the approach outlined in the previous section is used to decompose the total gross export flows, i.e. including intermediate goods. Already, a large body of literature exists on this topic, and therefore, we compare our approach with other decomposition approaches suggested in the literature, particularly the decomposition outlined in Koopman et al. (2014) (as before, referred to as KWW). We already compared our decompositions to KWW in the previous section, but here, we focus directly on gross export decomposition. In the next section, we outline the relationship with the results based on the hypothetical extraction method that has been presented as an alternative in Los et al. (2016). In the first subsection, we provide two decompositions of intermediate goods use that we then use for a decomposition of total bilateral gross export flows together with the results presented in the previous section.

5.1 Intermediate goods trade

The first way intermediate's domestic use and bilateral exports can be decomposed is to distinguish between domestic and foreign content (analogous to the decomposition presented in the previous section). The other decompositions (sink-source and sink-assembly) are not suited to the case of intermediary inputs because this would lead to double-counting problems. However, we propose an alternative by taking into account that intermediate use is a function of final goods demand in the Leontief demand driven model.

5.1.1 Domestic and foreign content of intermediate goods trade

We start by differentiating total bilateral gross exports into trade in final products and intermediates.[41] As already made clear, final products are products that are assembled in a country (directly and indirectly embodying value added from many sources) and are then domestically consumed or further exported to a country where these are absorbed (sink). Intermediaries are products that are further used for production purposes. Exports of these can either be - eventually after some further processing - used in the country of arrival or again further re-exported in the form of intermediates. These intermediary exports are the off-diagonal blocks in the transactions matrix $\tilde{\mathbf{Z}}$, whereas the diagonal blocks are the domestic use. Focusing on the traded goods and aggregating over the using industries results in $\tilde{\mathbf{E}} = \tilde{\mathbf{F}} + \tilde{\mathbf{Z}}_a$, where subscript a denotes that the transactions matrix is aggregated over using industries and thus of dimension $NC \times C$. Table 5.1 reports the corresponding numbers for the domestic use and trade of intermediates for the four country groups. The bilateral trade flows can be decomposed into the domestic and foreign

[41] Though we mostly refer to and discuss exports, it is important to note that - by definition - these constitute the imports of the respective partner countries.

Table 5.1: Gross trade matrix for final and intermediate goods

Importer \ Exporter	EU-28	China	USA	RoW	Total	EU-28	China	USA	RoW	Total
	Domestic use $\hat{\mathbf{Z}}$					Intermediates trade $\tilde{\mathbf{Z}}$				
EU-28	12,767	0	0	0	12,767	2,484	139	273	1,504	4,401
China	0	19,972	0	0	19,972	185	0	130	897	1,212
USA	0	0	12,164	0	12,164	324	66	0	871	1,261
RoW	0	0	0	27,308	27,308	1,256	1,169	987	3,053	6,465
Total	12,767	19,972	12,164	27,308	72,211	4,249	1,374	1,390	6,325	13,339

Note: Values in bn USD.
Source: WIOD Release 2016; own calculations.

contents analogous to Section 4 for final goods. The corresponding equation is[42]

$$\mathbf{Z}_a = \underbrace{(\bar{\mathbf{B}}\hat{\mathbf{Z}}_a + \breve{\mathbf{B}}\hat{\mathbf{Z}}_a) + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\hat{\mathbf{Z}}_a}_{\text{Domestic use}} + \underbrace{(\bar{\mathbf{B}}\tilde{\mathbf{Z}}_a + \breve{\mathbf{B}}\tilde{\mathbf{Z}}_a) + (\widehat{\mathbf{1}\tilde{\mathbf{B}}})\tilde{\mathbf{Z}}_a}_{\text{Exported intermediates}} \tag{5.1}$$

from which a decomposition into the domestic and foreign content of total bilateral gross exports follows immediately when being combined with equation (4.3). The numbers for bilateral final goods exports have been presented in the previous section; for completeness, the corresponding figures for intermediates are presented in Table 5.2.

Table 5.2: Value-added content of intermediate use

	EU-28	China	USA	RoW	Total	EU-28	China	USA	RoW	Total
	$\bar{\mathbf{B}}\hat{\mathbf{Z}}$					$\bar{\mathbf{B}}\tilde{\mathbf{Z}}$				
EU-28	10,386	0	0	0	10,386	1,643	96	188	1,043	2,970
China	0	17,273	0	0	17,273	152	0	105	752	1,009
USA	0	0	11,099	0	11,099	290	58	0	751	1,099
RoW	0	0	0	21,853	21,853	970	836	734	2,342	4,882
Total	10,386	17,273	11,099	21,853	60,611	3,055	990	1,026	4,888	9,960
	$\breve{\mathbf{B}}\hat{\mathbf{Z}}$					$\breve{\mathbf{B}}\tilde{\mathbf{Z}}$				
EU-28	36	0	0	0	36	15	1	2	7	25
China	0	109	0	0	109	2	0	1	7	11
USA	0	0	55	0	55	2	0	0	6	9
RoW	0	0	0	205	205	11	15	8	23	57
Total	36	109	55	205	405	30	16	11	44	101
	$(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\hat{\mathbf{Z}}$					$(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{Z}}$				
EU-28	2,343	0	0	0	2,343	826	41	84	454	1,405
China	0	2,590	0	0	2,590	31	0	24	138	193
USA	0	0	1,008	0	1,008	32	8	0	113	153
RoW	0	0	0	5,248	5,248	271	318	243	687	1,518
Total	2,343	2,590	1,008	5,248	11,189	1,160	368	351	1,391	3,269

Note: Values in bn USD.
Source: WIOD Release 2016; own calculations.

[42] In this case, aggregation over using industries would not be needed, however, it is convenient later when being combined with final goods exports.

5.1.2 Intermediate goods trade for domestic and foreign absorption

In the demand-driven Leontief model, the use of intermediates is endogenous, i.e. depending on final demand and technical coefficients. The transactions matrix can be expressed with input-output coefficients as $\mathbf{Z} = \mathbf{A}\hat{\mathbf{x}} = \hat{\mathbf{A}}\hat{\mathbf{x}} + \tilde{\mathbf{A}}\hat{\mathbf{x}}$. In the latter expression, the first matrix denotes the domestic use of intermediates, whereas the second matrix traded intermediates. Because $\mathbf{x} = \mathbf{LF1} = \mathbf{L}\hat{\mathbf{F}}\mathbf{1} + \mathbf{L}\tilde{\mathbf{F}}\mathbf{1}$ one can write

$$\mathbf{Z} = \underbrace{\hat{\mathbf{A}}(\widehat{\mathbf{L}\hat{\mathbf{F}}\mathbf{1}}) + \hat{\mathbf{A}}(\widehat{\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}})}_{\text{Domestic use}} + \underbrace{\tilde{\mathbf{A}}(\widehat{\mathbf{L}\hat{\mathbf{F}}\mathbf{1}}) + \tilde{\mathbf{A}}(\widehat{\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}})}_{\text{Traded intermediates}}$$

Focusing on traded intermediates, the first matrix denotes intermediate trade associated with demand on domestically assembled and consumed final products f^{rr}. This means that the assembly and source of the final product takes place in a country r. The assembly of this product requires intermediary inputs that are imported before the final assembly stage. The entries of this matrix, therefore, represent all bilateral intermediary flows associated with these needs for inputs (parts and components) to assemble the final product in the sink country. The second matrix includes intermediate trade associated with demand on products that are finally assembled in one country and exported to another country, i.e f^{rs}. Analogously to the above, the assembly of the exported final product requires intermediary inputs from other countries. The entries of this matrix therefore represent all bilateral intermediary flows associated with these needs for inputs (parts and components) to produce the final product in the assembly country before its being shipped to another country. Similar interpretations hold for the first two terms (domestic use), except that the intermediary products are sourced domestically. The corresponding numbers are presented in Table 5.3.[43] The panels on the left show the corresponding domestic flows. The upper panel on the right shows intermediate trade associated with the assembly of final products that are absorbed in the assembly country. The lower panel is intermediate trade associated with final goods exports.

5.2 Decomposition of bilateral gross exports

5.2.1 Decomposition

For a decomposition of gross export flows, we sum up matrix $\tilde{\mathbf{A}}(\widehat{\mathbf{L}\hat{\mathbf{F}}\mathbf{1}})$ over using industries, which results in a $NC \times C$ matrix denoted by $\tilde{\mathbf{Z}}_{\text{fdom}}$. Similarly, denote matrix $\tilde{\mathbf{A}}(\widehat{\mathbf{L}\hat{\mathbf{F}}\mathbf{1}})$ in this proper dimension as $\tilde{\mathbf{Z}}_{\text{fexp}}$. Using the split of the value-added multiplier matrix results in a decomposition of gross exports

[43]These are $NC \times NC$ matrices, which are summed up over country groups and industries.

Table 5.3: Intermediate flows by assembly/sink dimension

	EU-28	China	USA	RoW	Total	EU-28	China	USA	RoW	TotalÂ
	$\hat{\mathbf{A}}(\widehat{\mathbf{L}\hat{\mathbf{F}}\mathbf{1}})$					$\tilde{\mathbf{A}}(\widehat{\mathbf{L}\hat{\mathbf{F}}\mathbf{1}})$				
EU-28	10,253	0	0	0	10,253	1,738	118	250	1,279	3,386
China	0	17,132	0	0	17,132	128	0	118	737	983
USA	0	0	11,437	0	11,437	225	57	0	721	1,003
RoW	0	0	0	23,986	23,986	917	952	903	2,583	5,354
Total	10,253	17,132	11,437	23,986	62,808	3,008	1,127	1,271	5,320	10,726
	$\hat{\mathbf{A}}(\widehat{\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}})$					$\tilde{\mathbf{A}}(\widehat{\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}})$				
EU-28	2,514	0	0	0	2,514	747	21	23	225	1,015
China	0	2,840	0	0	2,840	57	0	13	160	229
USA	0	0	728	0	728	98	10	0	150	257
RoW	0	0	0	3,322	3,322	339	218	84	470	1,111
Total	2,514	2,840	728	3,322	9,404	1,240	248	119	1,006	2,613

Note: Values in bn USD.
Source: WIOD Release 2016; own calculations.

with nine terms

$$\begin{aligned}
\tilde{\mathbf{E}} \quad = \quad & \underbrace{\left[\bar{\mathbf{B}}\tilde{\mathbf{F}} + \breve{\mathbf{B}}\tilde{\mathbf{F}}\right] + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{F}}}_{\text{Final goods exports}} + \\
& \underbrace{\left[\bar{\mathbf{B}}\tilde{\mathbf{Z}}_{\text{fdom}} + \breve{\mathbf{B}}\tilde{\mathbf{Z}}_{\text{fdom}}\right] + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{Z}}_{\text{fdom}}}_{\text{Intermediates trade for 'assembly=sink'}} + \\
& \underbrace{\left[\bar{\mathbf{B}}\tilde{\mathbf{Z}}_{\text{fexp}} + \breve{\mathbf{B}}\tilde{\mathbf{Z}}_{\text{fexp}}\right] + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{Z}}_{\text{fexp}}}_{\text{Intermediates trade for 'assembly} \neq \text{sink'}}
\end{aligned} \tag{5.2}$$

This decomposition splits bilateral gross export flows into final goods exports, and the two types of intermediate trade explained above. Within these groups, gross exports are split into the two components of domestic content (pure and with multiple border crossings) and foreign content. The results of these calculations are presented in Table 5.4.[44] The panels in the first row present the terms in bilateral gross exports, whereas the panels in the second to fourth rows present the corresponding value-added flows differentiating between the purely domestic value-added content, the domestic content with multiply border crossings and the foreign content of the bilateral gross exports.

[44]Compared to the previous results, note that $\bar{\mathbf{B}}\tilde{\mathbf{Z}}_{\text{fdom}} + \bar{\mathbf{B}}\tilde{\mathbf{Z}}_{\text{fexp}} = \bar{\mathbf{B}}\tilde{\mathbf{Z}}$ as shown in Table 5.2.

Table 5.4: Decomposition of bilateral gross exports

	Final goods trade					Intermediates goods trade										Total goods
						Direct absorption					Re-exported					
	EU-28	China	USA	RoW	Total	EU-28	China	USA	RoW	Total	EU-28	China	USA	RoW	Total	
Total																
	$\tilde{\mathbf{F}}$					$\tilde{\mathbf{A}}(\widehat{\mathbf{L}\hat{\mathbf{F}}\mathbf{1}})$					$\tilde{\mathbf{A}}(\widehat{\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}})$					
EU-28	1,387	138	206	1,093	2,823	1,738	118	250	1,279	3,386	747	21	23	225	1,015	7,224
China	181	0	217	815	1,213	128	0	118	737	983	57	0	13	160	229	2,425
USA	130	46	0	491	666	225	57	0	721	1,003	98	10	0	150	257	1,927
RoW	520	285	595	1,208	2,607	917	952	903	2,583	5,354	339	218	84	470	1,111	9,072
Total	2,217	468	1,018	3,607	7,310	3,008	1,127	1,271	5,320	10,726	1,240	248	119	1,006	2,613	20,649
Pure domestic value added in gross exports																
	$\bar{\mathbf{B}}\tilde{\mathbf{F}}$					$\bar{\mathbf{B}}\tilde{\mathbf{A}}(\widehat{\mathbf{L}\hat{\mathbf{F}}\mathbf{1}})$					$\bar{\mathbf{B}}\tilde{\mathbf{A}}(\widehat{\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}})$					
EU-28	924	97	144	765	1,930	1,158	82	172	890	2,302	485	14	15	154	668	4,900
China	149	0	177	682	1,008	106	0	95	621	822	46	0	10	131	187	2,017
USA	113	38	0	430	581	202	49	0	624	875	88	8	0	128	224	1,679
RoW	380	201	408	905	1,894	714	689	673	1,990	4,066	256	147	61	351	816	6,776
Total	1,566	335	729	2,782	5,412	2,180	821	940	4,124	8,065	875	170	86	764	1,895	15,372
Multiple counted domestic value added in gross exports																
	$\breve{\mathbf{B}}\tilde{\mathbf{F}}$					$\breve{\mathbf{B}}\tilde{\mathbf{A}}(\widehat{\mathbf{L}\hat{\mathbf{F}}\mathbf{1}})$					$\breve{\mathbf{B}}\tilde{\mathbf{A}}(\widehat{\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}})$					
EU-28	8	1	2	6	17	10	1	1	6	18	5	0	0	1	6	42
China	2	0	2	8	12	1	0	1	6	8	1	0	0	2	2	23
USA	1	0	0	3	5	1	0	0	5	7	1	0	0	1	2	13
RoW	5	3	5	8	20	8	12	7	19	46	3	3	1	4	11	77
Total	16	5	9	25	55	21	13	10	35	79	9	3	1	8	22	156
Foreign content in bilateral gross exports																
	$(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{F}}$					$(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}(\widehat{\mathbf{L}\hat{\mathbf{F}}\mathbf{1}})$					$(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}(\widehat{\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}})$					
EU-28	454	40	60	321	875	569	35	77	383	1,064	257	6	7	70	341	2,280
China	30	0	37	126	193	21	0	22	110	153	10	0	2	27	40	386
USA	15	7	0	58	81	22	7	0	92	122	10	1	0	21	32	234
RoW	135	81	181	294	691	192	251	221	572	1,236	79	67	22	114	282	2,209
Total	634	128	279	799	1,840	805	293	319	1,158	2,575	355	75	32	233	695	5,109

Note: Values in bn USD.
Source: WIOD Release 2016; own calculations.

5.2.2 Gross export decomposition and value-added exports

For an in-depth understanding of the terms appearing in this decomposition, we relate them to the terms capturing value-added exports in equation (4.1), i.e.

$$[\bar{\mathbf{B}}\tilde{\mathbf{F}} + \breve{\mathbf{B}}\tilde{\mathbf{F}}] + \tilde{\mathbf{B}}\hat{\mathbf{F}} + \widetilde{\tilde{\mathbf{B}}\tilde{\mathbf{F}}} \tag{5.3}$$

This also allows us to relate this decomposition of gross exports to the KWW approach. Remember from Section 4 that the double-counting terms (KWW6 and KWW9) and the foreign value added in exports of intermediate goods (KWW8) have not yet derived. We first discuss these relationships in an intuitive way and provide detailed proofs in the next subsection. The first two terms in equation (5.2) or equation (5.3) belong to final goods exports (i.e. finally assembled in one country and finally absorbed in another country) and have been discussed extensively in Section 4. The third term in equation (5.2) captures the foreign content of a country's bilateral final goods exports and has been discussed above.

The third term in equation (5.3), $\tilde{\mathbf{B}}\hat{\mathbf{F}}$, denotes the intermediate products that are exported and finally assembled and absorbed in the sink country – characterised above as *[source*$^r \rightsquigarrow^{\forall c}$ *assembly*$^s \rightarrow$ *sink*^{s}*]* – and are therefore part of intermediate exports. This is related to equation (5.2) in the following way:

$$\bar{\mathbf{B}}\tilde{\mathbf{Z}}_{\text{fdom}}\mathbf{1} = \bar{\mathbf{B}}\tilde{\mathbf{A}}(\widehat{\mathbf{L}\hat{\mathbf{F}}\mathbf{1}})\mathbf{1} = \text{VAT6(=KWW2)*+VAT2(=KWW5)*} = \tilde{\mathbf{B}}\hat{\mathbf{F}}\mathbf{1} + \breve{\mathbf{B}}\hat{\mathbf{F}}\mathbf{1}$$

where * denotes that this only holds for a country's total gross exports. The term $\bar{\mathbf{B}}\tilde{\mathbf{Z}}_{\text{fdom}}$ first includes the domestic value added in intermediate goods exports absorbed by direct importers (KWW2), i.e. $\tilde{\mathbf{B}}\hat{\mathbf{F}}$. But, because these direct importers also include the source country of value added (which exports the intermediate but finally absorbs them again as the (re-)importer of value added), $\bar{\mathbf{B}}\tilde{\mathbf{Z}}_{\text{fdom}}$ also includes the 'domestic value added in intermediate goods exports re-imported as intermediate goods and finally absorbed at home (VAT2=KWW5), i.e. $\breve{\mathbf{B}}\hat{\mathbf{F}}$. Therefore, $\tilde{\mathbf{B}}\hat{\mathbf{F}}$ is part of value-added exports in equation (5.3), whereas the latter term $\breve{\mathbf{B}}\hat{\mathbf{F}}$ is not. However both constitute intermediary exports captured in equation (5.2). These two sides of value-added flows cannot be looked at in a bilateral way simultaneously and therefore only hold for a country's total gross exports (see formal proof in the next subsection).[45]

The fourth term in equation (5.3), $(\widetilde{\tilde{\mathbf{B}}\tilde{\mathbf{F}}})$, has been characterised as 'intermediate exports assembled in one country and sent as a final product to a third country', i.e. *[source*$^r \rightsquigarrow^{\forall c}$ *assembly*$^s \rightarrow$ *sink*^{t}*]*. One can show that this term is part of $\bar{\mathbf{B}}\tilde{\mathbf{Z}}_{\text{fexp}}\mathbf{1}$ (again for a country's total exports) in the following way:

$$\bar{\mathbf{B}}\tilde{\mathbf{Z}}_{\text{fexp}}\mathbf{1} = \bar{\mathbf{B}}\tilde{\mathbf{A}}(\widehat{\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}})\mathbf{1} = \text{VAT7(=KWW3)*+VAT3(=KWW4)*+VAT5*} = (\widetilde{\tilde{\mathbf{B}}\tilde{\mathbf{F}}})\mathbf{1} + (\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{F}}})\mathbf{1} + \breve{\mathbf{B}}\tilde{\mathbf{F}}\mathbf{1}$$

[45] Technically, it has already been shown that $\breve{\mathbf{B}}\hat{\mathbf{F}} = (\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{A}}\bar{\mathbf{L}}\hat{\mathbf{F}}})$, which has entries only at the diagonal (as constituting domestic absorption) and is not compatible with a bilateral gross exports decomposition.

Thus, domestic value added in intermediate goods exports re-exported to third countries (VAT7=KWW3), domestic value added in intermediate goods exports re-imported as final goods (VAT3=KWW4), and VAT5 (related to domestic multipliers with multiple border crossings) sum up to the intermediate trade to assemble a final product abroad which is shipped to a third country (including also the source country).

Further, we proof in the next subsection that double-counted intermediate exports originally produced at home (KWW6) are composed of all components related to the domestic multipliers with multiple border crossings in equation (5.2). Formally, this means

$$KWW6 = (\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{A}}\bar{\mathbf{L}}\hat{\mathbf{e}}^*})\mathbf{1} = \breve{\mathbf{B}}\tilde{\mathbf{F}} + \breve{\mathbf{B}}\tilde{\mathbf{Z}}_{\text{fdom}} + \breve{\mathbf{B}}\tilde{\mathbf{Z}}_{\text{fexp}}$$

And, finally, we proof in the next subsection that the foreign content of intermediate exports is related to the KWW approach in the following way (where we have to split the Leontief inverse into the submatrices $\mathbf{L} = \bar{\mathbf{L}} + \breve{\mathbf{L}} + \tilde{\mathbf{L}}$:

$$(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{Z}}_{\text{fdom}} + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{Z}}_{\text{fexp}} = (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}\mathbf{L}(\widehat{\hat{\mathbf{F}}\mathbf{1}}) + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}\mathbf{L}(\widehat{\tilde{\mathbf{F}}\mathbf{1}}) = \underbrace{(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}\bar{\mathbf{L}}(\widehat{\hat{\mathbf{F}}\mathbf{1}})}_{\text{KWW8}} + \underbrace{(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}(\breve{\mathbf{L}} + \tilde{\mathbf{L}})(\widehat{\hat{\mathbf{F}}\mathbf{1}}) + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}\mathbf{L}(\widehat{\tilde{\mathbf{F}}\mathbf{1}})}_{\text{KWW9}}$$

which includes KWW8 (Foreign value added in exports of intermediate goods) and KWW9 (double-counted intermediate exports originally produced abroad).

5.2.3 Summary

To summarise, in this section, we presented a decomposition of bilateral gross exports in value-added terms and indicated the relationships to the KWW decomposition for total exports. Formally, this results in the following statement

$$\begin{aligned}
\tilde{\mathbf{E}}\mathbf{1} \quad = \quad & \underbrace{\overbrace{[\bar{\mathbf{B}}\tilde{\mathbf{F}} + \breve{\mathbf{B}}\tilde{\mathbf{F}}]\mathbf{1}}^{\hat{\mathbf{B}}\tilde{\mathbf{F}}\mathbf{1}}}_{\text{KWW1}} + \overbrace{\underbrace{\bar{\mathbf{B}}\tilde{\mathbf{Z}}_{\text{fdom}}\mathbf{1}}_{\text{KWW2+KWW5}}}^{\tilde{\mathbf{B}}\hat{\mathbf{F}}\mathbf{1}+\breve{\mathbf{B}}\hat{\mathbf{F}}\mathbf{1}} + \underbrace{\overbrace{\left[\bar{\mathbf{B}}\tilde{\mathbf{Z}}_{\text{fexp}} - \breve{\mathbf{B}}\tilde{\mathbf{F}}\right]\mathbf{1}}^{(\widetilde{\tilde{\mathbf{B}}\tilde{\mathbf{F}}})+(\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{F}}})}}_{\text{KWW3+KWW4}} + \underbrace{\overbrace{[\breve{\mathbf{B}}\tilde{\mathbf{F}} + \breve{\mathbf{B}}\tilde{\mathbf{Z}}_{\text{fdom}} + \breve{\mathbf{B}}\tilde{\mathbf{Z}}_{\text{fexp}}]\mathbf{1}}^{(\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{A}}\bar{\mathbf{L}}\hat{\mathbf{e}}*})}}_{\text{KWW6}} + \\
& \underbrace{(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{F}}\mathbf{1}}_{\text{KWW7}} + \underbrace{\overbrace{(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{Z}}_{\text{fdom},1}\mathbf{1}}^{(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}\bar{\mathbf{L}}(\widehat{\hat{\mathbf{F}}\mathbf{1}})}}_{\text{KWW8}} + \underbrace{\overbrace{(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{Z}}_{\text{fdom},2}\mathbf{1} + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{Z}}_{\text{fexp}}\mathbf{1}}^{(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}(\breve{\mathbf{L}}+\tilde{\mathbf{L}})(\widehat{\hat{\mathbf{F}}\mathbf{1}})+(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}\mathbf{L}(\widehat{\tilde{\mathbf{F}}\mathbf{1}})}}_{\text{KWW9}}
\end{aligned}$$

where the term $(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{Z}}_{\text{fdom},2}\mathbf{1}$ is split into two terms (see technical details below). In the next subsection, we proof these relationships.

5.3 Relationship to KWW

5.3.1 Representation of KWW and a more detailed bilateral gross exports decomposition

To show the relation to the decomposition provided in KWW, we first represent this in matrix notation in a bilateral way (though it is intended to decompose a country's total exports, which will become clear soon).[46] The KWW decomposition (for details see Appendix Section B) is given by

$$\mathbf{K1} = \underbrace{\hat{\mathbf{B}}\tilde{\mathbf{F}}\mathbf{1} + \tilde{\mathbf{B}}\hat{\mathbf{F}}\mathbf{1} + \widetilde{\tilde{\mathbf{B}}\tilde{\mathbf{F}}\mathbf{1}}}_{\text{Value added exports}} + \underbrace{\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{F}}\mathbf{1}} + \tilde{\mathbf{B}}\widehat{\tilde{\mathbf{A}}\bar{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}}_{\text{Value added re-imports}} + \underbrace{\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{A}}\bar{\mathbf{L}}}\hat{\mathbf{e}}^{*}\mathbf{1}}_{\text{DCdom}} + \underbrace{(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{F}}\mathbf{1} + \widehat{(\mathbf{1}'\tilde{\mathbf{B}})}\tilde{\mathbf{A}}\bar{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}_{\text{Foreign content}} + \underbrace{\widehat{(\mathbf{1}'\tilde{\mathbf{B}})}\tilde{\mathbf{A}}\bar{\mathbf{L}}\hat{\mathbf{e}}^{*}\mathbf{1}}_{\text{DCfor}}$$

We refer to the nine terms as listed in this equation as KWW1 to KWW9. The KWW decomposition is derived for a country's total gross exports, but expressing it in the notation used in this paper, we show in the Appendix Section B that some of the matrices - according to these manipulations - are diagonal (which, by definition, cannot be the case for bilateral gross exports). This happens for KWW4, KWW5 and KWW6. A closer inspection shows that this concerns the re-imported value added via final and intermediate goods, which in terms of gross exports are off-diagonal elements. KWW6 would be the domestic double-counted term. For further inspection of the relationship to KWW, one can decompose the parts for intermediate trade even further by splitting the Leontief matrix into the three components $\mathbf{L} = \bar{\mathbf{L}} + \breve{\mathbf{L}} + \tilde{\mathbf{L}}$. Applying these results in a decomposition of twenty-one terms:

$$\begin{aligned}
\tilde{\mathbf{E}} \quad = \quad & \underbrace{\left[\bar{\mathbf{B}}\tilde{\mathbf{F}} + \breve{\mathbf{B}}\tilde{\mathbf{F}}\right] + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{F}}}_{\text{Final goods exports}} + \\
& \underbrace{\left[\bar{\mathbf{B}}\tilde{\mathbf{A}}\big((\bar{\mathbf{L}} + \widehat{\breve{\mathbf{L}} + \tilde{\mathbf{L}}})\hat{\mathbf{F}}\mathbf{1}\big) + \breve{\mathbf{B}}\tilde{\mathbf{A}}\big((\bar{\mathbf{L}} + \widehat{\breve{\mathbf{L}} + \tilde{\mathbf{L}}})\hat{\mathbf{F}}\mathbf{1}\big)\right] + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}\big((\bar{\mathbf{L}} + \widehat{\breve{\mathbf{L}} + \tilde{\mathbf{L}}})\hat{\mathbf{F}}\mathbf{1}\big)}_{\text{Directly absorbed intermediates}} + \\
& \underbrace{\left[\bar{\mathbf{B}}\tilde{\mathbf{A}}\big((\bar{\mathbf{L}} + \widehat{\breve{\mathbf{L}} + \tilde{\mathbf{L}}})\tilde{\mathbf{F}}\mathbf{1}\big) + \breve{\mathbf{B}}\tilde{\mathbf{A}}\big((\bar{\mathbf{L}} + \widehat{\breve{\mathbf{L}} + \tilde{\mathbf{L}}})\tilde{\mathbf{F}}\mathbf{1}\big)\right] + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}\big((\bar{\mathbf{L}} + \widehat{\breve{\mathbf{L}} + \tilde{\mathbf{L}}})\tilde{\mathbf{F}}\mathbf{1}\big)}_{\text{Re-exported intermediates}}
\end{aligned}$$

We argue that the following relationships between the bilateral gross exports decomposition outlined here to the KWW approach hold and proof these (and therefore also the statements in the previous subsection) below when using some of these additional terms. Formally, this can be summarised in the

[46] Formally, it holds that $\mathbf{K1} = \tilde{\mathbf{E}}\mathbf{1}$.

following equation:

$$
\begin{aligned}
\tilde{\mathbf{E}} \;=\; & \underbrace{\overbrace{\left[\bar{\mathbf{B}}\tilde{\mathbf{F}} + \breve{\mathbf{B}}\tilde{\mathbf{F}}\right]}^{\text{KWW1}}}_{\text{DVA in final goods exports}} \quad + \qquad\qquad (5.4)\\
& \underbrace{\Big[\overbrace{\bar{\mathbf{B}}\tilde{\mathbf{A}}\big(\widehat{\mathbf{L}\hat{\mathbf{F}}\mathbf{1}}\big)}^{\text{(KWW2+KWW5)}^*} + \overbrace{\breve{\mathbf{B}}\tilde{\mathbf{A}}\big(\widehat{\mathbf{L}\hat{\mathbf{F}}\mathbf{1}}\big)}^{\text{1st part of (KWW6}-\breve{\mathbf{B}}\tilde{\mathbf{F}})^*}\Big]}_{\text{DVA in directly absorbed intermediates}} + \\
& \underbrace{\Big[\overbrace{\bar{\mathbf{B}}\tilde{\mathbf{A}}\big(\widehat{\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}}\big)}^{\text{(KWW3+KWW4+}\breve{\mathbf{B}}\tilde{\mathbf{F}})^*} + \overbrace{\breve{\mathbf{B}}\tilde{\mathbf{A}}\big(\widehat{\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}}\big)}^{\text{2nd part of (KWW6}-\breve{\mathbf{B}}\tilde{\mathbf{F}})^*}\Big]}_{\text{DVA in re-exported intermediates}} + \\
& \underbrace{\underbrace{\overbrace{(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{F}}}^{\text{KWW7}}}_{\text{... final goods exports}} + \underbrace{\overbrace{(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}(\widehat{\bar{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}})}^{\text{KWW8}} + \overbrace{(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}\big(\widehat{(\breve{\mathbf{L}}+\tilde{\mathbf{L}})\hat{\mathbf{F}}\mathbf{1}}\big)}^{\text{1st Part of KWW9}}}_{\text{... directly absorbed intermediates}} + \underbrace{\overbrace{(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}\big(\widehat{\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}}\big)}^{\text{2nd part of KWW9}}}_{\text{... re-exported intermediates}}}_{\text{Foreign value added in ...}}
\end{aligned}
$$

For the terms marked with $*$, the equivalence only holds for a country's total exports (i.e. not in a bilateral way), in line with the KWW approach being not genuinely bilateral. In the following, we discuss these relations in detail and prove the respective equivalence for each of the terms, as indicated in equation (5.4). Specifically, the double-counted term (KWW6 or DCdom) is related to the country's trade of intermediates that cross borders multiple times, but are assembled into the final product at home, i.e. $\breve{\mathbf{B}}\tilde{\mathbf{F}}$. Details are provided in the following. Table 5.5 shows the resulting numbers based on the numerical example where some of the terms appearing in KWW are summed up according to their relationship to the decomposition of gross export flows in this paper (the nine individual terms are reported in Appendix Table B.1). For simplicity, we denote the ten terms in equation (5.4) with E1 to E10 and compare with KWW1 to KWW9. In cases where only the row sums coincide, these are marked with *.

5.3.2 Technical details and proofs

KWW1: The first obvious difference is that in our approach, the *domestic value added in direct final goods exports (KWW1)* is split into the 'pure' term and the part that cross borders multiple times though the final assembly stage is in the country of origin, i.e. $\hat{\mathbf{B}}\tilde{\mathbf{F}} = \bar{\mathbf{B}}\tilde{\mathbf{F}} + \breve{\mathbf{B}}\tilde{\mathbf{F}}$. Though this might look like a pure definitional issue, we will see below that the term $\breve{\mathbf{B}}\tilde{\mathbf{F}}$ plays a crucial role in the way this approach is related to the double-counting term KWW6 (DomDC).

(KWW2+KWW5):* Next, we show that KWWs *Domestic value added in intermediate goods exports absorbed by direct importers (KWW2)* and the *Domestic value added in intermediate goods exports re-imported as intermediate goods and finally absorbed at home (KWW5)* sum up to the first term in the

Table 5.5: Comparison to KWW

	Source-assembly-sink decomposition					KWW decomposition				
	EU-28	China	USA	RoW	Total	EU-28	China	USA	RoW	Total
	E1: DVA in FG exports - pure					KWW1				
EU-28	924	97	144	765	1,930	932	98	146	771	1,947
China	149	0	177	682	1,008	151	0	180	690	1,020
USA	113	38	0	430	581	114	38	0	433	586
RoW	380	201	408	905	1,894	385	204	412	913	1,914
Total	1,566	335	729	2,782	5,412	1,582	340	738	2,807	5,467
	E2: DVA in FG exports - multiple					Part of KWW1				
EU-28	8	1	2	6	17					
China	2	0	2	8	12					
USA	1	0	0	3	5					
RoW	5	3	5	8	20					
Total	16	5	9	25	55					
	E3: DVA in dir. abs. II exports - pure					(KWW2+KWW5)*				
EU-28	1,158	82	172	890	2,302	982	133	231	957	2,302
China	106	0	95	621	822	110	38	116	557	822
USA	202	49	0	624	875	170	71	61	573	875
RoW	714	689	673	1,990	4,066	660	723	732	1,950	4,066
Total	2,180	821	940	4,124	8,065	1,923	966	1,140	4,036	8,065
	E5: DVA in re-exp. II - pure					(KWW3+KWW4 +$\breve{\mathbf{B}}\tilde{\mathbf{F}}$)*				
EU-28	485	14	15	154	668	297	40	69	262	668
China	46	0	10	131	187	53	21	33	80	187
USA	88	8	0	128	224	62	13	61	87	224
RoW	256	147	61	351	816	238	59	124	396	816
Total	875	170	86	764	1,895	650	133	287	825	1,895
	E4+E6: DVA in II - mult.					(KWW6$-\breve{\mathbf{B}}\tilde{\mathbf{F}}$)*				
EU-28	15	1	2	7	25	34	-1	-2	-6	25
China	2	0	1	7	11	-2	23	-2	-8	11
USA	2	0	0	6	9	-1	-0	13	-3	9
RoW	11	15	8	23	57	-5	-3	-5	70	57
Total	30	16	11	44	101	26	18	5	52	101
	E7: FVA in FG exports					KWW7				
EU-28	454	40	60	321	875	454	40	60	321	875
China	30	0	37	126	193	30	0	37	126	193
USA	15	7	0	58	81	15	7	0	58	81
RoW	135	81	181	294	691	135	81	181	294	691
Total	634	128	279	799	1,840	634	128	279	799	1,840
	E8: FVA in dir. abs. II exports (Part 1)					KWW8				
EU-28	355	31	69	295	749	355	31	69	295	749
China	14	0	19	81	115	14	0	19	81	115
USA	13	6	0	70	90	13	6	0	70	90
RoW	123	210	198	423	955	123	210	198	423	955
Total	505	248	287	868	1,908	505	248	287	868	1,908
	E9+E10: FVA in II exports (Part 2)					KWW9				
EU-28	472	10	15	159	656	472	10	15	159	656
China	17	0	4	57	78	17	0	4	57	78
USA	19	2	0	43	64	19	2	0	43	64
RoW	148	108	44	264	564	148	108	44	264	564
Total	655	120	64	523	1,361	655	120	64	523	1,361

Note: Values in bn USD.
Source: WIOD Release 2016; own calculations.

directly absorbed intermediates in equation (5.4), $\bar{\mathbf{B}}\tilde{\mathbf{A}}(\widehat{\mathbf{L}\hat{\mathbf{F}}\mathbf{1}})$, but this is the case only for total exports (i.e. not bilateral exports).[47] Formally, one has to show that

$$\tilde{\mathbf{B}}\hat{\mathbf{F}}\mathbf{1} + \widehat{\tilde{\mathbf{B}}\tilde{\mathbf{A}}\bar{\mathbf{L}}\hat{\mathbf{F}}}\mathbf{1} = \bar{\mathbf{B}}\tilde{\mathbf{A}}\mathbf{L}\hat{\mathbf{F}}\mathbf{1}$$

where the lhs is the row sum of the two terms appearing in KWW, and the rhs is the term in this decomposition.

Proof: In Section 4 we have already shown that $\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{A}}\bar{\mathbf{L}}\hat{\mathbf{F}}} = \breve{\mathbf{B}}\hat{\mathbf{F}}$, i.e. KWW5 equals the value-added flows that represent value added from the source country and that, after multiple international border crossings, is assembled into the final product in the original source country and then shipped (as part of the final product) to other countries. Compared to KWW, these are separated out explicitly. Therefore, this allows us to rewrite the above expression as $\tilde{\mathbf{B}}\hat{\mathbf{F}}\mathbf{1} + \breve{\mathbf{B}}\hat{\mathbf{F}}\mathbf{1} = \bar{\mathbf{B}}\tilde{\mathbf{A}}\mathbf{L}\hat{\mathbf{F}}\mathbf{1}$. Noting that $\tilde{\mathbf{B}} + \breve{\mathbf{B}} = \mathbf{B} - \bar{\mathbf{B}}$, it follows that $(\mathbf{B} - \bar{\mathbf{B}})\hat{\mathbf{F}}\mathbf{1} = \bar{\mathbf{B}}\tilde{\mathbf{A}}\mathbf{L}\hat{\mathbf{F}}\mathbf{1}$. This equality holds if $\mathbf{B} - \bar{\mathbf{B}} = \bar{\mathbf{B}}\tilde{\mathbf{A}}\mathbf{L}$ (because both sides are post-multiplied with the same vector $\hat{\mathbf{F}}\mathbf{1}$). Pre-multiplying with the inverse of the diagonalized value-added coefficient vector results in $\mathbf{L} - \bar{\mathbf{L}} = \bar{\mathbf{L}}\tilde{\mathbf{A}}\mathbf{L}$ or $\mathbf{L} = \bar{\mathbf{L}}\tilde{\mathbf{A}}\mathbf{L} + \bar{\mathbf{L}}$. The latter expression follows from the property of inverse matrices. □

(KWW3+KWW4):* The next terms considered are the *Domestic value added in intermediate goods exports re-exported to third countries (KWW3)* and *Domestic value added in intermediate goods exports re-imported as final goods (KWW4)*, which corresponds to a term in our decomposition when subtracting the 'complex domestic value-added part' and considering total exports, i.e.

$$\widetilde{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}\mathbf{1} + \widehat{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}\mathbf{1} = \bar{\mathbf{B}}\tilde{\mathbf{A}}(\widehat{\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}})\mathbf{1} - \breve{\mathbf{B}}\tilde{\mathbf{F}}\mathbf{1}$$

Proof: The diagonal and off-diagonal terms on the lhs can be summed together, to which the last term on the rhs can be added. Because only the row sums are considered; the first term on the rhs can be rewritten, which results in

$$\tilde{\mathbf{B}}\tilde{\mathbf{F}}\mathbf{1} + \breve{\mathbf{B}}\tilde{\mathbf{F}}\mathbf{1} = (\tilde{\mathbf{B}} + \breve{\mathbf{B}})\tilde{\mathbf{F}}\mathbf{1} = \bar{\mathbf{B}}\tilde{\mathbf{A}}\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}$$

The lhs can now be expressed as the difference between the diagonal of the global and the domestic matrices, i.e. $(\tilde{\mathbf{B}} + \breve{\mathbf{B}}) = \mathbf{B} - \bar{\mathbf{B}}$. Pre-multiplying with the inverse of the diagonalised value-added coefficients matrix finally results in $\mathbf{L} - \bar{\mathbf{L}} = \bar{\mathbf{L}}\tilde{\mathbf{A}}\mathbf{L}$ and follows from the property of inverse matrices. □

[47] In this respect, it is important to note that KWW5 - in the representation provided in Appendix Section B - shifts terms, or more exactly, the 're-imports of value added' to the diagonal that - by definition - cannot be part of gross-trade flows (or exports). For this reason, KWW is not a bilateral approach, and therefore, this only holds for a country's total exports.

KWW6:* Next, we show that the term capturing *Double-counting home (KWW6)* equals the three terms capturing a country's trade with multiple border crossings for total exports, i.e.[48]

$$(\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{A}}\bar{\mathbf{L}}\hat{\mathbf{e}}^*})\mathbf{1} = (\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{A}}\bar{\mathbf{L}}})\hat{\mathbf{e}}^*\mathbf{1} = \breve{\mathbf{B}}\tilde{\mathbf{F}}\mathbf{1} + \left[\breve{\mathbf{B}}\tilde{\mathbf{A}}(\widehat{\mathbf{L}\hat{\mathbf{F}}\mathbf{1}}) + \breve{\mathbf{B}}\tilde{\mathbf{A}}(\widehat{\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}})\right]\mathbf{1}$$

PROOF: This can be easily shown by inserting the property of inverse matrices (pre-multiplied with the diagonalized value-added coefficient vector) for the diagonal elements, i.e. $\hat{\mathbf{B}} - \bar{\mathbf{B}} = (\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{A}}\bar{\mathbf{L}}})$ or on the lhs and adding up the last two terms on the rhs $(\hat{\mathbf{B}} - \bar{\mathbf{B}})\hat{\mathbf{e}}^*\mathbf{1} = \breve{\mathbf{B}}\tilde{\mathbf{F}}\mathbf{1} + \breve{\mathbf{B}}\tilde{\mathbf{A}}(\widehat{\mathbf{L}\mathbf{F}\mathbf{1}})\mathbf{1}$. Using $\breve{\mathbf{B}} = \hat{\mathbf{B}} - \bar{\mathbf{B}}$ and $\tilde{\mathbf{A}}(\widehat{\mathbf{L}\mathbf{F}\mathbf{1}}) = \tilde{\mathbf{A}}\hat{\mathbf{x}}$ results in[49] $\breve{\mathbf{B}}\hat{\mathbf{e}}^*\mathbf{1} = \breve{\mathbf{B}}\tilde{\mathbf{F}}\mathbf{1} + \breve{\mathbf{B}}(\tilde{\mathbf{A}}\hat{\mathbf{x}})\mathbf{1} = \breve{\mathbf{B}}\tilde{\mathbf{F}}\mathbf{1} + \breve{\mathbf{B}}\tilde{\mathbf{Z}}\mathbf{1} = \breve{\mathbf{B}}\tilde{\mathbf{E}}\mathbf{1}$. □

This result shows that the KWW domestic double-counting term equals the value added generated in a country, crossing borders multiple times and returning back for assembly for both final goods and intermediate exports.

Turning to the remaining items in the decomposition presented in equation (5.4), the foreign value-added content of bilateral gross trade is described by three terms $(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{F}} + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}(\widehat{\mathbf{L}\hat{\mathbf{F}}\mathbf{1}}) + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}(\widehat{\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}})$.

KWW7: The first term, $(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{F}}$ equals the *foreign value-added content of final goods trade (KWW7).*

KWW8: To show the relationship of the remaining terms, the second term has to be rearranged by splitting the Leontief inverse matrix into $\mathbf{L} = \bar{\mathbf{L}} + \breve{\mathbf{L}} + \tilde{\mathbf{L}}$ as already indicated above. The second term can then be written as

$$(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}(\widehat{\mathbf{L}\hat{\mathbf{F}}\mathbf{1}}) = (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}\big(\widehat{(\bar{\mathbf{L}} + \breve{\mathbf{L}} + \tilde{\mathbf{L}})\hat{\mathbf{F}}\mathbf{1}}\big) = (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}(\widehat{\bar{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}) + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}(\widehat{\breve{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}) + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}(\widehat{\tilde{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}})$$

The first term on the rhs can be rewritten as $(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}(\widehat{\bar{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}) = (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}\bar{\mathbf{L}}\hat{\mathbf{F}}$ and therefore equals the *foreign value added in exports of intermediate goods (KWW8).* The remaining two terms on the rhs are part of KWW9, as shown next.

KWW9: Thus, we finally have to show that the sum of the remaining terms equals the *double-counted intermediate exports originally produced abroad (KWW9)*, i.e.

$$(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}\bar{\mathbf{L}}\hat{\mathbf{e}}^* = (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}(\widehat{\breve{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}) + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}(\widehat{\tilde{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}) + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}(\widehat{\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}})$$

[48] Note that the term $\breve{\mathbf{B}}\tilde{\mathbf{F}}\mathbf{1}$ is subtracted from KWW6, whereas it has been added to KWW4+KWW5 above; for ease of explanation, this term has shifted to the rhs in both expressions.

[49] This equality can also be proved by inserting for $\hat{\mathbf{e}}^*\mathbf{1}$ on the lhs $\breve{\mathbf{B}}\tilde{\mathbf{F}}\mathbf{1} + \breve{\mathbf{B}}\tilde{\mathbf{A}}\mathbf{L}\mathbf{F}\mathbf{1} = \breve{\mathbf{B}}\tilde{\mathbf{F}}\mathbf{1} + \breve{\mathbf{B}}\tilde{\mathbf{A}}(\widehat{\mathbf{L}\mathbf{F}\mathbf{1}})\mathbf{1}$

Proof: Inserting for $\hat{\mathbf{e}}^*$ on the lhs results in

$$(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}\bar{\mathbf{L}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}}) + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}\bar{\mathbf{L}}(\widehat{\tilde{\mathbf{A}}(\widehat{\mathbf{LF1}})\mathbf{1}}) = (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}(\widehat{\breve{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}) + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}(\widehat{\tilde{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}) + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}(\widehat{\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}})$$

which can be slightly simplified to

$$(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}\bar{\mathbf{L}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}}) + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}\bar{\mathbf{L}}(\widehat{\tilde{\mathbf{A}}\mathbf{LF1}}) = (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}(\widehat{\breve{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}) + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}(\widehat{\tilde{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}) + (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}(\widehat{\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}})$$

Because all terms are pre-multiplied with the same matrices, $(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})$ and $\tilde{\mathbf{A}}$, one can reduce this to

$$\bar{\mathbf{L}}(\widehat{\tilde{\mathbf{F}}\mathbf{1}}) + \bar{\mathbf{L}}(\widehat{\tilde{\mathbf{A}}\mathbf{LF1}}) = (\widehat{\breve{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}) + (\widehat{\tilde{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}) + (\widehat{\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}})$$

Noting that these are all diagonal matrices, the lhs can be rewritten as

$$(\widehat{\bar{\mathbf{L}}\tilde{\mathbf{F}}\mathbf{1}}) + (\widehat{\bar{\mathbf{L}}\tilde{\mathbf{A}}\mathbf{LF1}}) = (\widehat{\breve{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}) + (\widehat{\tilde{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}) + (\widehat{\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}})$$

Applying the property of inverse matrices of the diagonal elements (see Appendix Section C) on the lhs and replacing $\breve{\mathbf{L}} = \mathbf{L} - \tilde{\mathbf{L}} - \bar{\mathbf{L}}$ on the rhs, the expression can be simplified as

$$\begin{aligned}
(\widehat{\bar{\mathbf{L}}\tilde{\mathbf{F}}\mathbf{1}}) + (\widehat{\mathbf{LF1}}) - (\widehat{\bar{\mathbf{L}}\mathbf{F1}}) &= (\widehat{\mathbf{L}\hat{\mathbf{F}}\mathbf{1}}) - (\widehat{\tilde{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}) - (\widehat{\bar{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}) + (\widehat{\tilde{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}) + (\widehat{\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}}) \\
(\widehat{\mathbf{LF1}}) - (\widehat{\bar{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}) &= (\widehat{\mathbf{L}\hat{\mathbf{F}}\mathbf{1}}) - (\widehat{\tilde{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}) - (\widehat{\bar{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}) + (\widehat{\tilde{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}) + (\widehat{\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}}) \\
(\widehat{\mathbf{LF1}}) &= (\widehat{\mathbf{L}\hat{\mathbf{F}}\mathbf{1}}) - (\widehat{\tilde{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}) + (\widehat{\tilde{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}) + (\widehat{\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}}) \\
(\widehat{\mathbf{LF1}}) &= (\widehat{\mathbf{L}\hat{\mathbf{F}}\mathbf{1}}) + (\widehat{\mathbf{L}\tilde{\mathbf{F}}\mathbf{1}}) = (\widehat{\mathbf{LF1}})
\end{aligned}$$

showing the equivalence of the expression in the decomposition derived here and the term KWW9. □

This equation indicates that KWW9 ('double counting foreign') can be expressed as

$$(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}\bar{\mathbf{L}}\hat{\mathbf{e}}^* = (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}(\widehat{\mathbf{LF1}}) - (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}(\widehat{\bar{\mathbf{L}}\hat{\mathbf{F}}\mathbf{1}}) = (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{Z}} - (\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{A}}\bar{\mathbf{L}}\hat{\mathbf{F}}$$

i.e. the foreign content in bilateral intermediary exports minus the foreign content of intermediary exports that are finally assembled and absorbed in the home country (i.e. where assembly and sink takes place).

6 Decomposition of value chains using the hypothetical extraction method

A related – though, in the literature, often considered an alternative – approach is the hypothetical extraction method. Therefore, in this section, we argue that the approach outlined so far can also be interpreted along the lines of the hypothetical extraction method as suggested in Los et al. (2016), Los and Timmer (2018), and extended in Borin and Mancini (2019) for considering global value chains.[50] Using this method, one sets certain elements of the coefficients matrix to $\mathbf{A}$ zero, calculates the corresponding Leontief inverse, and calculates the respective indicator (e.g. value-added exports). Finally, the difference between this 'hypothetical' result and the original result can be calculated. Los et al. (2016) show that, depending on the elements extracted, parts of the KWW decomposition can be calculated.[51] In the following two subsections, we first argue that the results achieved so far can be interpreted as a special case of the hypothetical extraction method. Second, we then argue that the hypothetical extraction method is a useful tool in specifying the exact nature of what one defines as a value chain and what aspects one likes to consider. We provide an example and separate out intra-regional value-added flows within the framework suggested in this paper, focusing on the sink-source approach.

6.1 A special case

The approach outlined in the previous sections can be interpreted as a special case of the hypothetical extraction method. Specifically, in the above approach, the perturbed coefficients matrix is $\hat{\mathbf{A}}$, i.e. the matrix where all non-domestic coefficients are set to zero, and are therefore 'hypothetically extracted'. The related (local) Leontief inverse has been denoted by $\bar{\mathbf{L}}$. This matrix therefore captures the purely domestic chains with the other flows being removed (or set to zero).[52] Taking into account the other parts of the Leontief inverse, i.e. $\check{\mathbf{L}}$ and the off-diagonal blocks $\tilde{\mathbf{L}}$, which sum up to the global Leontief, and the split of the final demand block, allows us to track all value-added flows as discussed in Section 4 and Section 5. Thus, in essence, the approach presented in the previous sections can be interpreted as a special case of the hypothetical extraction method, i.e. with a special perturbation of the coefficients matrix where all off-diagonal elements of matrix $\mathbf{A}$ are set to zero and traced separately.

[50] For a general introduction to the hypothetical extraction method, see Miller and Blair (2009).

[51] However, this approach is criticised because when applied to various value chains, the "adding-up property" is violated. See the discussions in Koopman et al. (2016) and Los and Timmer (2018).

[52] Note that if all but the block-diagonal elements of the coefficients matrix are set to zero, the Leontief inverse is block-diagonal, and these blocks equal the local (or domestic) inverse.

6.2 Refining the source-sink decomposition applying the hypothetical extraction method

6.2.1 Outline of extended decomposition

It is straightforward to argue that the above calculations are doable and meaningful (particularly when tracing the remaining flows in the way outlined above), but need careful interpretations. We exemplify this by providing an extension of the framework presented so far. Specifically, we additionally set the coefficients of the global coefficients matrix for non-EU countries to zero, which separates the pure intra-EU flows of the respective value chains. By including additional matrices that capture the missing flows, we circumvent the 'adding up' problem - as discussed in Koopman et al. (2016) and Los and Timmer (2018) - by adding another layer to the decomposition. By doing so, we keep all flows, thus maintaining the level of global value added, and specifically argue that all appearing terms allow for a neat interpretation.[53] This example therefore focuses on the size and patterns of intra-EU value-added flows and trace all other flows separately.

6.2.2 Multiplier matrices capturing intra-EU flows separately

For doing so, we define a perturbed coefficients matrix where all extra-EU flows are set to zero. Extra-EU flows, for example, mean flows between China and the US, but also flows from Germany to the US. This matrix is denoted by $\ddot{\mathbf{A}}$. The associated Leontief inverse (gross output multiplier) matrix is denoted by $\ddot{\mathbf{L}}$. We denote the value-added multiplier matrix as $\ddot{\mathbf{B}} = \hat{\mathbf{v}}\ddot{\mathbf{L}}$ correspondingly. Using these matrices, we extend the decomposition of the multiplier matrices, which results in

$$\mathbf{B} = \bar{\mathbf{B}} + \underbrace{(\hat{\ddot{\mathbf{B}}} - \bar{\mathbf{B}}) + (\hat{\mathbf{B}} - \hat{\ddot{\mathbf{B}}})}_{\breve{\mathbf{B}}=\hat{\mathbf{B}}-\bar{\mathbf{B}}} + \underbrace{\tilde{\ddot{\mathbf{B}}} + (\tilde{\mathbf{B}} - \tilde{\ddot{\mathbf{B}}})}_{\tilde{\mathbf{B}}} \tag{6.1}$$

The corresponding decomposition of the gross and value-added multipliers is presented in Table 6.1 (which sum up to the figures already presented in Table 4.1). These are now used to provide an extended decomposition of the source-sink matrix.

[53] Of course, there is a huge number of combinations to set specific elements of the coefficients matrix to zero, and this is just an example.

Table 6.1: Multiplier decomposition using hypothetical extraction

	Gross output				Value added			
	EU-28	China	USA	RoW	EU-28	China	USA	RoW
	$\mathbf{L}$				$\mathbf{B}$			
EU-28	1.873	0.038	0.037	0.098	0.905	0.016	0.016	0.041
China	0.038	2.729	0.029	0.087	0.011	0.877	0.008	0.025
USA	0.034	0.017	1.678	0.050	0.018	0.008	0.925	0.025
RoW	0.156	0.258	0.123	2.029	0.066	0.099	0.052	0.909
	$\bar{\mathbf{L}}$				$\bar{\mathbf{B}}$			
EU-28	1.632	0.000	0.000	0.000	0.807	0.000	0.000	0.000
China	0.000	2.710	0.000	0.000	0.000	0.872	0.000	0.000
USA	0.000	0.000	1.669	0.000	0.000	0.000	0.921	0.000
RoW	0.000	0.000	0.000	1.807	0.000	0.000	0.000	0.816
	$\hat{\ddot{\mathbf{L}}} - \bar{\mathbf{L}}$				$\hat{\ddot{\mathbf{B}}} - \bar{\mathbf{B}}$			
EU-28	0.006	0.000	0.000	0.000	0.003	0.000	0.000	0.000
China	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
USA	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
RoW	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	$\hat{\mathbf{L}} - \hat{\ddot{\mathbf{L}}}$				$\hat{\mathbf{B}} - \hat{\ddot{\mathbf{B}}}$			
EU-28	0.001	0.000	0.000	0.000	0.001	0.000	0.000	0.000
China	0.000	0.019	0.000	0.000	0.000	0.005	0.000	0.000
USA	0.000	0.000	0.009	0.000	0.000	0.000	0.004	0.000
RoW	0.000	0.000	0.000	0.017	0.000	0.000	0.000	0.007
	$\tilde{\ddot{\mathbf{L}}}$				$\tilde{\ddot{\mathbf{B}}}$			
EU-28	0.221	0.000	0.000	0.000	0.090	0.000	0.000	0.000
China	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
USA	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
RoW	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	$\tilde{\mathbf{L}} - \tilde{\ddot{\mathbf{L}}}$				$\tilde{\mathbf{B}} - \tilde{\ddot{\mathbf{B}}}$			
EU-28	0.011	0.038	0.037	0.098	0.005	0.016	0.016	0.041
China	0.038	0.000	0.029	0.087	0.011	0.000	0.008	0.025
USA	0.034	0.017	0.000	0.050	0.018	0.008	0.000	0.025
RoW	0.156	0.258	0.123	0.206	0.066	0.099	0.052	0.087

Note: Values in bn USD.
Source: WIOD Release 2016; own calculations.

6.2.3 An extended decomposition

Inserting this expression source-sink matrix $\mathbf{T} = \mathbf{BF}$ and rearranging (analogous to Section 4) results in

$$\begin{aligned}
\mathbf{T} \quad = \quad & \overbrace{\bar{\mathbf{B}}\hat{\mathbf{F}} + \underbrace{(\hat{\ddot{\mathbf{B}}} - \bar{\mathbf{B}})\hat{\mathbf{F}} + (\hat{\mathbf{B}} - \hat{\ddot{\mathbf{B}}})\hat{\mathbf{F}}}_{\breve{\mathbf{B}}\hat{\mathbf{F}}}}^{\hat{\mathbf{B}}\hat{\mathbf{F}}} + \underbrace{\widehat{\tilde{\ddot{\mathbf{B}}}\tilde{\mathbf{F}}} + \widehat{(\tilde{\mathbf{B}} - \tilde{\ddot{\mathbf{B}}})\tilde{\mathbf{F}}}}_{\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}} + \\
& \overbrace{\bar{\mathbf{B}}\tilde{\mathbf{F}} + \underbrace{(\hat{\ddot{\mathbf{B}}} - \bar{\mathbf{B}})\tilde{\mathbf{F}} + (\hat{\mathbf{B}} - \hat{\ddot{\mathbf{B}}})\tilde{\mathbf{F}}}_{\breve{\mathbf{B}}\tilde{\mathbf{F}}}}^{\hat{\mathbf{B}}\tilde{\mathbf{F}}} + \underbrace{\tilde{\ddot{\mathbf{B}}}\hat{\mathbf{F}} + (\tilde{\mathbf{B}} - \tilde{\ddot{\mathbf{B}}})\hat{\mathbf{F}}}_{\tilde{\mathbf{B}}\hat{\mathbf{F}}} + \underbrace{\widetilde{\tilde{\ddot{\mathbf{B}}}\tilde{\mathbf{F}}} + \widetilde{(\tilde{\mathbf{B}} - \tilde{\ddot{\mathbf{B}}})\tilde{\mathbf{F}}}}_{\widetilde{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}} \qquad (6.2)
\end{aligned}$$

As one can see, this is a straightforward extension of the equation (4.1). This is arranged so that the first line comprises domestic consumption, and the second line comprises value-added exports. In Appendix

Section D, we present the various matrices in detail; here, we give an intuitive interpretation following Section 4.

Domestic consumption: Table 6.2 reports the corresponding figures for our numerical example for domestic absorption. The first element, $\bar{\mathbf{B}}\hat{\mathbf{F}}$, is unchanged and denotes the purely domestic value-added flows Â Â Â *[source^r $\circlearrowleft^r$Â assembly^r $\rightarrow$ sink^r]* (see Section 4). The second and third elements in the decomposition sum up to $\breve{\mathbf{B}}\hat{\mathbf{F}}$ and constitute the flows for domestic absorption with multiple border crossings characterised above as *[source^r $\rightsquigarrow^{\forall c}$ assembly^r $\rightarrow$ sink^r]*. This termÂ is further split into its pure intra-EU flows and all the remaining flows. In detail, the term $(\hat{\ddot{\mathbf{B}}} - \bar{\mathbf{B}})\hat{\mathbf{F}}$ captures a country's flows with multiple intra-EU border crossings (all other flaws are set to zero, and the pure intra-country flows are subtracted), which therefore can be characterised as *[source$^{r\in EU}$ $\rightsquigarrow^{\circlearrowleft\forall c\in EU(r\neq c)}$ assembly$^{r\in EU}$ $\rightarrow$ sink$^{r\in EU}$]*. The other term, $(\hat{\mathbf{B}} - \hat{\ddot{\mathbf{B}}})\hat{\mathbf{F}}$, captures all 'complex flows' excluding these 'complex' intra-EU flows. For discussing them, it is easier to distinguish between the EU and the non-EU countries. For the EU countries, this term includes the flows with countries outside the EU. These flows can therefore be characterised as *[source$^{r\in EU}$ $\rightsquigarrow^{\forall c\setminus\circlearrowleft\forall c\in EU(c\neq r)}$ assembly$^{r\in EU}$ $\rightarrow$ sink$^{r\in EU}$]*. Technically, all flows excluding the pure intra-country flows are built into matrix $\hat{\mathbf{B}}$ (which includes the diagonalized blocks of the global multiplier matrix) from which the pure intra-EU flows are subtracted. For the non-EU countries, these flows can be characterised as *[source$^{r\notin EU}$ $\rightsquigarrow^{\forall c}$ assembly$^{r\notin EU}$ $\rightarrow$ sink$^{r\notin EU}$]*. Note that these flows also include flows between non-EU countries and EU countries.

Next, the term $\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}$ shows the re-imports of value added that have been characterised in Section 4 as *[source^r $\rightsquigarrow^{\forall c}$ assembly$^{c\neq r}$ $\rightarrow$ sink^r]* . Again, these are split up into the pure intra-EU flows and those including extra-EU flows similar to above. Accordingly, the first term $\widehat{\tilde{\ddot{\mathbf{B}}}\tilde{\mathbf{F}}}$ captures all intra-EU flows and therefore can be characterised as *[source$^{r\in EU}$ $\rightsquigarrow^{\circlearrowleft\forall c\in EU}$ assembly$^{c(\neq r)\in EU}$ $\rightarrow$ sink^r]*. The difference to the above is that assembly takes place in another country in the EU than the source country to which, however, the final product is shipped.

The final term $\widehat{(\tilde{\mathbf{B}} - \tilde{\ddot{\mathbf{B}}})}\tilde{\mathbf{F}}$ then captures all remaining flows. Again, for discussion, it is easier to distinguish between EU and non-EU countries. For the EU countries, these flows are characterised as *[source$^{r\in EU}$ $\rightsquigarrow^{\forall c\setminus\circlearrowleft\forall c\in EU}$ assembly^c $\rightarrow$ sink$^{r\in EU}$]*. This now also includes extra-EU flows (and not only pure intra-EU flows). Finally, for the non-EU countries, these comprise flows *[source$^{r\notin EU}$ $\rightsquigarrow^{\forall c}$ assembly^c $\rightarrow$ sink$^{r\notin EU}$]*. Again, these chains also include EU countries.

Value-added exports: The second line in equation (6.2) decomposes value-added export flows. Technical details are presented in Appendix Section D, whereas here, an intuitive explanation is given. Table 6.3 presents the empirical results. The first term $\bar{\mathbf{B}}\tilde{\mathbf{F}}$ captures the pure domestic chains of products that are exported (as final products), i.e. *[source^r $\circlearrowleft^r$Â assembly^r $\rightarrow$ sink^c]* as already discussed in Section 4. The second term $\breve{\mathbf{B}}\tilde{\mathbf{F}}$ includes the chains for exports with multiple border crossings characterised as

Table 6.2: TiVA decomposition of domestic consumption using the hypothetical extraction method for intra-EU flows

	Gross output					Value added				
	EU-28	China	USA	RoW	Total	EU-28	China	USA	RoW	Total
	Total					Total				
EU-28	12,516	0	0	0	12,516	6	0	0	0	6
China	0	8,441	0	0	8,441	0	38	0	0	38
USA	0	0	15,860	0	15,860	0	0	61	0	61
RoW	0	0	0	23,776	23,776	0	0	0	140	140
Total	12,516	8,441	15,860	23,776	60,593	6	38	61	140	246
	$\bar{\mathbf{L}}\hat{\mathbf{F}}$					$\bar{\mathbf{B}}\hat{\mathbf{F}}$				
EU-28	12,445	0	0	0	12,445	33	0	0	0	33
China	0	8,382	0	0	8,382	0	0	0	0	0
USA	0	0	15,737	0	15,737	0	0	0	0	0
RoW	0	0	0	23,511	23,511	0	0	0	0	0
Total	12,445	8,382	15,737	23,511	60,075	33	0	0	0	33
	$(\hat{\bar{\mathbf{L}}} - \bar{\mathbf{L}})\hat{\mathbf{F}}$					$(\hat{\bar{\mathbf{B}}} - \bar{\mathbf{B}})\hat{\mathbf{F}}$				
EU-28	24	0	0	0	24	8	0	0	0	8
China	0	0	0	0	0	0	21	0	0	21
USA	0	0	0	0	0	0	0	61	0	61
RoW	0	0	0	0	0	0	0	0	125	125
Total	24	0	0	0	24	8	21	61	125	214
	$(\hat{\mathbf{L}} - \hat{\bar{\mathbf{L}}})\hat{\mathbf{F}}$					$(\hat{\mathbf{B}} - \hat{\bar{\mathbf{B}}})\hat{\mathbf{F}}$				
EU-28	14	0	0	0	14	6	0	0	0	6
China	0	137	0	0	137	0	38	0	0	38
USA	0	0	128	0	128	0	0	61	0	61
RoW	0	0	0	355	355	0	0	0	140	140
Total	14	137	128	355	634	6	38	61	140	246
	$\widehat{\tilde{\bar{\mathbf{L}}}}\tilde{\mathbf{F}}$					$\widehat{\tilde{\bar{\mathbf{B}}}}\tilde{\mathbf{F}}$				
EU-28	81	0	0	0	81	33	0	0	0	33
China	0	0	0	0	0	0	0	0	0	0
USA	0	0	0	0	0	0	0	0	0	0
RoW	0	0	0	0	0	0	0	0	0	0
Total	81	0	0	0	81	33	0	0	0	33
	$\widehat{(\tilde{\mathbf{L}} - \tilde{\bar{\mathbf{L}}})}\tilde{\mathbf{F}}$					$\widehat{(\tilde{\mathbf{B}} - \tilde{\bar{\mathbf{B}}})}\tilde{\mathbf{F}}$				
EU-28	17	0	0	0	17	8	0	0	0	8
China	0	76	0	0	76	0	21	0	0	21
USA	0	0	134	0	134	0	0	61	0	61
RoW	0	0	0	336	336	0	0	0	125	125
Total	17	76	134	336	563	8	21	61	125	214

Note: Values in bn USD.
Source: WIOD Release 2016; own calculations.

[source^r $\rightsquigarrow^{\circlearrowleft\forall c}$ $\hat{A}$ assembly^r $\to$ sink^c] in Section 4, which is now split into two terms. Here, the intra-EU chains are characterised as $(\hat{\bar{\mathbf{B}}} - \bar{\mathbf{B}})\hat{\mathbf{F}}$ *[source$^{r\in EU}$ $\rightsquigarrow^{\forall c\in EU}$ assembly$^{r\in EU}$ $\to$ sink^c]*. Note, whereas the production chains are intra-EU chains, the final product might be exported outside the EU.[54] The second part $(\hat{\mathbf{B}} - \hat{\bar{\mathbf{B}}})\hat{\mathbf{F}}$ captures the flows with multiple border crossings including all countries. For the EU countries, these are characterised as *[source$^{r\in EU}$ $\rightsquigarrow^{\forall c\setminus\rightsquigarrow^{c\in EU}}$ assembly$^{r\in EU}$ $\to$ sink^c]*, where again, final

[54]One could further split whether these are exported to another EU or non-EU country by appropriately splitting the final demand matrix.

products can be shipped within the EU or to non-EU countries. For a non-EU country, the characterisation is given by *[*$source^{r\notin EU} \rightsquigarrow^{\forall c} assembly^{r\notin EU} \rightarrow sink^{c}$*]*. These chains involve both EU and non-EU countries. Again, the final destination country can be either an EU or non-EU country.[55]

The third term is $\tilde{\mathbf{B}}\hat{\mathbf{F}}$ and captures the flows with assembly taking place in the country of final absorption, *[*$source^{r} \rightsquigarrow^{\circlearrowleft\forall c} assembly^{c} \rightarrow sink^{c}$*]*. This is split first into the pure intra-EU flows as captured by $\ddot{\tilde{\mathbf{B}}}\hat{\mathbf{F}}$, which therefore can be characterised as *[*$source^{r\in EU} \rightsquigarrow^{\circlearrowleft\forall c\in EU} assembly^{c\in EU} \rightarrow sink^{c\in EU}$*]*. The second term is given by $(\tilde{\mathbf{B}} - \ddot{\tilde{\mathbf{B}}})\hat{\mathbf{F}}$. For EU countries, these capture the chains including other countries in the world with final assembly and absorption taking place in another country (other than the source of value added), i.e.Â *[*$source^{r\in EU}\hat{A} \quad \hat{A} \quad \rightsquigarrow^{\circlearrowleft\forall c\in EU\backslash\rightsquigarrow^{c\in EU}} assembly^{c} \rightarrow sink^{c}$*]*. For non-EU countries, these include all chains (including EU and non-EU countries) with assembly and absorption taking place in another country, i.e. *[*$source^{r\notin EU} \rightsquigarrow^{\circlearrowleft\forall c} assembly^{c} \rightarrow sink^{c}$*]*.

Finally, the last term $\widetilde{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}$: *[*$source^{r} \rightsquigarrow^{\circlearrowleft\forall c} assembly^{p} \rightarrow sink^{c}$*]* captures the chains with the country of final absorption being different from the country of final assembly and the source country (of value added). These first include the flows included as $\widetilde{\ddot{\tilde{\mathbf{B}}}\tilde{\mathbf{F}}}$. The respective flows are purely intra-EU, with the final assembly of the product taking place in an EU country (other than the source of value added), and are absorbed in a country outside the EU, or *[*$source^{r} \rightsquigarrow^{\circlearrowleft\forall c\in EU} assembly^{p\in EU} \rightarrow sink^{c}$*]*. The remaining flows are included as $\widetilde{(\tilde{\mathbf{B}} - \ddot{\tilde{\mathbf{B}}})\tilde{\mathbf{F}}}$, and for the EU country, these are characterised as *[*$source^{r} \rightsquigarrow^{\circlearrowleft\forall c\backslash\rightsquigarrow^{c\in EU}} assembly^{p\notin EU} \rightarrow sink^{c}$*]* whereas for the non-EU countries, these are characterised as *[*$source^{r} \rightsquigarrow^{\circlearrowleft\forall c} assembly^{p} \rightarrow sink^{c}$*]*.

6.2.4 Summary

We have extended the decomposition by applying hypothetical extraction, highlighting the role of pure intra-EU value chains. This was done in a framework that captures all value-added flows in the global economy. It goes without saying that any other country groups might be distinguished in a similar way, allowing for alternative definitions of value chains one would like to consider. Further, some extensions are generally possible (e.g. to split the absorbing countries into EU and non-EU countries). The example given should show that such extensions allow for neat interpretations of the matrices involved (see also the discussion and technical details in Appendix Section D). More generally, it is argued that the hypothetical extraction method and the approaches of decomposing value-added flows extensively discussed in the previous sections can be reconciled and are not competing concepts. Using this method and the corresponding decompositions of the multiplier matrices for defining the value chains one would like to consider can also be applied to the other concepts, like a decomposition of the source-assembly matrix or gross exports decomposition.

[55] Again, by appropriately splitting the final demand matrix, one could consider different export destination country groups.

Table 6.3: TiVA decomposition of value-added exports using the hypothetical extraction method for intra-EU flows

	Gross output					Value added				
	EU-28	China	USA	RoW	Total	EU-28	China	USA	RoW	Total
	Total					Total				
EU-28	2,132	270	444	1,983	4,829	905	0	0	0	905
China	312	0	327	1,319	1,958	0	0	0	0	0
USA	346	122	0	1,089	1,557	0	0	0	0	0
RoW	1,278	983	1,263	2,986	6,510	0	0	0	0	0
Total	4,068	1,375	2,034	7,377	14,854	905	0	0	0	905
	$\bar{\mathbf{L}}\tilde{\mathbf{F}}$					$\bar{\mathbf{B}}\tilde{\mathbf{F}}$				
EU-28	924	97	144	765	1,930	47	133	231	957	1,367
China	149	0	177	682	1,008	110	0	116	557	783
USA	113	38	0	430	581	170	71	0	573	814
RoW	380	201	408	905	1,894	660	723	732	1,810	3,926
Total	1,566	335	729	2,782	5,412	988	927	1,079	3,896	6,890
	$(\hat{\bar{\mathbf{L}}} - \bar{\mathbf{L}})\tilde{\mathbf{F}}$					$(\hat{\bar{\mathbf{B}}} - \bar{\mathbf{B}})\tilde{\mathbf{F}}$				
EU-28	7	1	1	5	15	196	21	29	153	398
China	0	0	0	0	0	0	0	0	0	0
USA	0	0	0	0	0	0	0	0	0	0
RoW	0	0	0	0	0	0	0	0	0	0
Total	7	1	1	5	15	196	21	29	153	398
	$(\hat{\mathbf{L}} - \hat{\bar{\mathbf{L}}})\tilde{\mathbf{F}}$					$(\hat{\mathbf{B}} - \hat{\bar{\mathbf{B}}})\tilde{\mathbf{F}}$				
EU-28	1	0	0	1	3	52	18	39	103	212
China	2	0	2	8	12	51	0	31	72	154
USA	1	0	0	3	5	61	13	0	83	157
RoW	5	3	5	8	20	233	56	119	263	671
Total	9	4	7	20	40	397	87	189	521	1,194
	$\tilde{\bar{\mathbf{L}}}\hat{\mathbf{F}}$					$\tilde{\bar{\mathbf{B}}}\hat{\mathbf{F}}$				
EU-28	2,186	0	0	0	2,186	905	0	0	0	905
China	0	0	0	0	0	0	0	0	0	0
USA	0	0	0	0	0	0	0	0	0	0
RoW	0	0	0	0	0	0	0	0	0	0
Total	2,186	0	0	0	2,186	905	0	0	0	905
	$(\tilde{\mathbf{L}} - \tilde{\bar{\mathbf{L}}})\hat{\mathbf{F}}$					$(\tilde{\mathbf{B}} - \tilde{\bar{\mathbf{B}}})\hat{\mathbf{F}}$				
EU-28	112	319	549	2,261	3,241	47	133	231	957	1,367
China	392	0	439	1,878	2,708	110	0	116	557	783
USA	328	147	0	1,169	1,644	170	71	0	573	814
RoW	1,547	1,899	1,748	4,312	9,507	660	723	732	1,810	3,926
Total	2,379	2,365	2,736	9,620	17,100	988	927	1,079	3,896	6,890
	$\widetilde{\tilde{\bar{\mathbf{L}}}}\tilde{\mathbf{F}}$					$\widetilde{\tilde{\bar{\mathbf{B}}}}\tilde{\mathbf{F}}$				
EU-28	495	53	73	379	1,000	196	21	29	153	398
China	0	0	0	0	0	0	0	0	0	0
USA	0	0	0	0	0	0	0	0	0	0
RoW	0	0	0	0	0	0	0	0	0	0
Total	495	53	73	379	1,000	196	21	29	153	398
	$(\tilde{\mathbf{L}} - \widetilde{\tilde{\bar{\mathbf{L}}}})\tilde{\mathbf{F}}$					$(\tilde{\mathbf{B}} - \widetilde{\tilde{\bar{\mathbf{B}}}})\tilde{\mathbf{F}}$				
EU-28	125	44	93	247	510	52	18	39	103	212
China	183	0	111	263	557	51	0	31	72	154
USA	119	25	0	166	310	61	13	0	83	157
RoW	584	143	315	661	1,703	233	56	119	263	671
Total	1,012	212	519	1,337	3,080	397	87	189	521	1,194

Note: Values in bn USD.
Source: WIOD Release 2016; own calculations.

7 Conclusions

This paper contributes to the existing literature in several ways. First, it introduces a straightforward approach based on simple matrix algebra and input-output analysis to calculate various value chain indicators. Most of them are well-known from the existing literature. The approach is to split various matrices (like the coefficients matrix, the Leontief inverse, the final demand matrix) into diagonal and off-diagonal blocks. Simple matrix multiplications then result in the various indicators, allowing for intuitive interpretations. Thus, second, this approach can be interpreted in a straightforward way by differentiating between the source country (where value added is generated and accounted for), the assembly country (the country where the final product is assembled, i.e. after this stage, no value added is added, and it goes straight to final consumption), and the sink country (the country where the final product is absorbed). A specific role in this discussion is played by the difference between the multiplier for a country calculated based on the global Leontief inverse (i.e. the block-diagonal elements of this matrix) and the 'domestic' multipliers (e.g. derived from a coefficient matrix only including the block-diagonal elements reflecting intra-country flows). This allows us, third, to provide a novel decomposition of bilateral gross exports resulting in nine components of value-added trade flows. This decomposition does not include double-counting terms because intermediate flows are traced back to final demand levels. In a further extension, even more detailed decompositions can be achieved. Fourth, it is shown how the gross export decomposition developed in this paper is related to the KWW approach. All terms, or a combination of them, can be aligned with the KWW decomposition, although some of them only for a country's total exports (because the KWW approach is genuinely derived from total exports) and sheds light on the nature of the 'double-counting terms'. Technically, the relations between this approach and the KWW approach are proved by applying the 'property of inverse matrices'. Finally, fifth, it is discussed how the 'hypothetical extraction method' can be used to add an additional layer to the decomposition framework promoted here. As an example, it is shown how this can be used to differentiate value-added flows across various countries (e.g. intra-EU-28 flows) and value-added flows globally.

In general, this paper adds to the literature by providing an alternative methodological framework leading to a novel decomposition of gross export flows and value-added flows in the global economy. Intuitive interpretations are provided distinguishing three stages of value-added flows in the global economy: source, assembly, and sink. The approach can be further used in several ways. First, the next step is to apply this method at the industry dimension. Related to this, second, one can use the hypothetical extraction method for a refined way of defining and characterising specific value chains of interest and to study their structures in the global economy. Finally, third, some of the results can be used to provide even more detailed decompositions of bilateral gross exports and study their relation to other approaches from the literature.

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A The power expansion of the global Leontief matrix

A.1 Power expansion and decomposition

Formally the Leontief inverse results from the power expansion of the coefficients matrix, i.e.

$$\mathbf{L} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \dots = (\mathbf{I} - \mathbf{A})^{-1}$$

Using the notation introduced above this can be split into its domestic (diagonal) and international (off-diagonal) elements, i.e.

$$\mathbf{L} = \mathbf{I} + (\hat{\mathbf{A}} + \tilde{\mathbf{A}}) + (\hat{\mathbf{A}} + \tilde{\mathbf{A}})^2 + (\hat{\mathbf{A}} + \tilde{\mathbf{A}})^3 + \dots = (\mathbf{I} - \mathbf{A})^{-1}$$

which can be reformulated as

$$\mathbf{L} = (\mathbf{I}+\hat{\mathbf{A}}+\hat{\mathbf{A}}^2+\hat{\mathbf{A}}^3+\dots)+\tilde{\mathbf{A}}+(\hat{\mathbf{A}}\tilde{\mathbf{A}}+\tilde{\mathbf{A}}\hat{\mathbf{A}}+\tilde{\mathbf{A}}^2)+(\hat{\mathbf{A}}\tilde{\mathbf{A}}\hat{\mathbf{A}}+\tilde{\mathbf{A}}\hat{\mathbf{A}}^2+\tilde{\mathbf{A}}^2\hat{\mathbf{A}}+\hat{\mathbf{A}}^2\tilde{\mathbf{A}}+\hat{\mathbf{A}}\tilde{\mathbf{A}}^2+\tilde{\mathbf{A}}\hat{\mathbf{A}}\tilde{\mathbf{A}}+\tilde{\mathbf{A}}^3)+\dots$$

The first term in brackets on the rhs constitutes the 'domestic' inverse $\bar{\mathbf{L}}$. From the remaining part the diagonal elements constitute $\breve{\mathbf{L}}$ whereas the off-diagonal elements are $\tilde{\mathbf{L}}$. For three countries this becomes

$$\begin{pmatrix} l^{11} & l^{12} & l^{13} \\ l^{21} & l^{22} & l^{23} \\ l^{31} & l^{32} & l^{33} \end{pmatrix} = \begin{pmatrix} \bar{l}^{1\circlearrowright 1} & 0 & 0 \\ 0 & \bar{l}^{2\circlearrowright 2} & 0 \\ 0 & 0 & \bar{l}^{3\circlearrowright 3} \end{pmatrix} + \begin{pmatrix} \breve{l}^{1\leftrightsquigarrow 1} & 0 & 0 \\ 0 & \breve{l}^{2\leftrightsquigarrow 2} & 0 \\ 0 & 0 & \breve{l}^{3\leftrightsquigarrow 3} \end{pmatrix} + \begin{pmatrix} 0 & l^{1\leftrightsquigarrow 2} & l^{1\leftrightsquigarrow 3} \\ l^{2\leftrightsquigarrow 1} & 0 & l^{2\leftrightsquigarrow 3} \\ l^{3\leftrightsquigarrow 1} & l^{3\leftrightsquigarrow 2} & 0 \end{pmatrix}$$

A.2 Detailed outline

For the case of three countries this power expansion looks like

$$\begin{aligned} \begin{pmatrix} l^{11} & l^{12} & l^{13} \\ l^{21} & l^{22} & l^{23} \\ l^{31} & l^{32} & l^{33} \end{pmatrix} = & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} a^{11} & a^{12} & a^{13} \\ a^{21} & a^{22} & a^{23} \\ a^{31} & a^{32} & a^{33} \end{pmatrix} + \\ & \begin{pmatrix} a^{11} & a^{12} & a^{13} \\ a^{21} & a^{22} & a^{23} \\ a^{31} & a^{32} & a^{33} \end{pmatrix} \begin{pmatrix} a^{11} & a^{12} & a^{13} \\ a^{21} & a^{22} & a^{23} \\ a^{31} & a^{32} & a^{33} \end{pmatrix} + \\ & \begin{pmatrix} a^{11} & a^{12} & a^{13} \\ a^{21} & a^{22} & a^{23} \\ a^{31} & a^{32} & a^{33} \end{pmatrix} \begin{pmatrix} a^{11} & a^{12} & a^{13} \\ a^{21} & a^{22} & a^{23} \\ a^{31} & a^{32} & a^{33} \end{pmatrix} \begin{pmatrix} a^{11} & a^{12} & a^{13} \\ a^{21} & a^{22} & a^{23} \\ a^{31} & a^{32} & a^{33} \end{pmatrix} + \dots \end{aligned}$$

Doing the matrix calculations this becomes

$$
\begin{aligned}
\begin{pmatrix} l^{11} & l^{12} & l^{13} \\ l^{21} & l^{22} & l^{23} \\ l^{31} & l^{32} & l^{33} \end{pmatrix} = & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} a^{11} & a^{12} & a^{13} \\ a^{21} & a^{22} & a^{23} \\ a^{31} & a^{32} & a^{33} \end{pmatrix} + \\
& \begin{pmatrix} a^{11}a^{11} + a^{12}a^{21} + a^{13}a^{31} & a^{11}a^{12} + a^{12}a^{22} + a^{13}a^{32} & a^{11}a^{13} + a^{12}a^{23} + a^{13}a^{33} \\ a^{21}a^{11} + a^{22}a^{21} + a^{23}a^{31} & a^{21}a^{12} + a^{22}a^{22} + a^{23}a^{32} & a^{21}a^{13} + a^{22}a^{23} + a^{23}a^{33} \\ a^{31}a^{11} + a^{32}a^{21} + a^{33}a^{31} & a^{31}a^{12} + a^{32}a^{22} + a^{33}a^{32} & a^{31}a^{13} + a^{32}a^{23} + a^{33}a^{33} \end{pmatrix} + \\
& \begin{pmatrix}
\begin{matrix}(a^{11}a^{11} + a^{12}a^{21} + a^{13}a^{31})a^{11} + \\ (a^{11}a^{12} + a^{12}a^{22} + a^{13}a^{32})a^{21} + \\ (a^{11}a^{13} + a^{12}a^{23} + a^{13}a^{33})a^{31}\end{matrix} & \begin{matrix}(a^{11}a^{11} + a^{12}a^{21} + a^{13}a^{31})a^{12} + \\ (a^{11}a^{12} + a^{12}a^{22} + a^{13}a^{32})a^{22} + \\ (a^{11}a^{13} + a^{12}a^{23} + a^{13}a^{33})a^{32}\end{matrix} & \begin{matrix}(a^{11}a^{11} + a^{12}a^{21} + a^{13}a^{31})a^{13} + \\ (a^{11}a^{12} + a^{12}a^{22} + a^{13}a^{32})a^{23} + \\ (a^{11}a^{13} + a^{12}a^{23} + a^{13}a^{33})a^{33}\end{matrix} \\
\begin{matrix}(a^{21}a^{11} + a^{22}a^{21} + a^{23}a^{31})a^{11} + \\ (a^{21}a^{12} + a^{22}a^{22} + a^{23}a^{32})a^{21} + \\ (a^{21}a^{13} + a^{22}a^{23} + a^{23}a^{33})a^{31}\end{matrix} & \begin{matrix}(a^{21}a^{11} + a^{22}a^{21} + a^{23}a^{31})a^{12} + \\ (a^{21}a^{12} + a^{22}a^{22} + a^{23}a^{32})a^{22} + \\ (a^{21}a^{13} + a^{22}a^{23} + a^{23}a^{33})a^{32}\end{matrix} & \begin{matrix}(a^{21}a^{11} + a^{22}a^{21} + a^{23}a^{31})a^{13} + \\ (a^{21}a^{12} + a^{22}a^{22} + a^{23}a^{32})a^{23} + \\ (a^{21}a^{13} + a^{22}a^{23} + a^{23}a^{33})a^{33}\end{matrix} \\
\begin{matrix}(a^{31}a^{11} + a^{32}a^{21} + a^{33}a^{31})a^{11} + \\ (a^{31}a^{12} + a^{32}a^{22} + a^{33}a^{32})a^{21} + \\ (a^{31}a^{13} + a^{32}a^{23} + a^{33}a^{33})a^{31}\end{matrix} & \begin{matrix}(a^{31}a^{11} + a^{32}a^{21} + a^{33}a^{31})a^{12} + \\ (a^{31}a^{12} + a^{32}a^{22} + a^{33}a^{32})a^{22} + \\ (a^{31}a^{13} + a^{32}a^{23} + a^{33}a^{33})a^{32}\end{matrix} & \begin{matrix}(a^{31}a^{11} + a^{32}a^{21} + a^{33}a^{31})a^{13} + \\ (a^{31}a^{12} + a^{32}a^{22} + a^{33}a^{32})a^{23} + \\ (a^{31}a^{13} + a^{32}a^{23} + a^{33}a^{33})a^{33}\end{matrix}
\end{pmatrix} + \dots
\end{aligned} \tag{A.1}
$$

Considering only the first three steps, i.e. $\mathbf{L} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^2 + \dots$ the split of the Leontief matrix used in text is given by

$$
\begin{aligned}
\begin{pmatrix} l^{11} & l^{12} & l^{13} \\ l^{21} & l^{22} & l^{23} \\ l^{31} & l^{32} & l^{33} \end{pmatrix} = & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} a^{11} & 0 & 0 \\ 0 & a^{22} & 0 \\ 0 & 0 & a^{33} \end{pmatrix} + \begin{pmatrix} a^{11}a^{11} & 0 & 0 \\ 0 & a^{22}a^{22} & 0 \\ 0 & 0 & a^{33}a^{33} \end{pmatrix} + \begin{pmatrix} a^{11}a^{11}a^{11} & 0 & 0 \\ 0 & a^{22}a^{22}a^{22} & 0 \\ 0 & 0 & a^{33}a^{33}a^{33} \end{pmatrix} + \dots + \\
& \begin{pmatrix} a^{12}a^{21} + a^{13}a^{31} & 0 & 0 \\ 0 & a^{21}a^{12} + a^{23}a^{32} & 0 \\ 0 & 0 & a^{31}a^{13} + a^{32}a^{23} \end{pmatrix} + \\
& \begin{pmatrix}
\begin{matrix}(a^{12}a^{21} + a^{13}a^{31})a^{11} + \\ (a^{11}a^{12} + a^{12}a^{22} + a^{13}a^{32})a^{21} + \\ (a^{11}a^{13} + a^{12}a^{23} + a^{13}a^{33})a^{31}\end{matrix} & 0 & 0 \\
0 & \begin{matrix}(a^{21}a^{11} + a^{22}a^{21} + a^{23}a^{31})a^{12} + \\ (a^{21}a^{12} + a^{23}a^{32})a^{22} + \\ (a^{21}a^{13} + a^{22}a^{23} + a^{23}a^{33})a^{32}\end{matrix} & 0 \\
0 & 0 & \begin{matrix}(a^{31}a^{11} + a^{32}a^{21} + a^{33}a^{31})a^{13} + \\ (a^{31}a^{12} + a^{32}a^{22} + a^{33}a^{32})a^{23} + \\ (a^{31}a^{13} + a^{32}a^{23} +)a^{33}\end{matrix}
\end{pmatrix} + \dots + \\
& \begin{pmatrix} 0 & a^{12} & a^{13} \\ a^{21} & 0 & a^{23} \\ a^{31} & a^{32} & 0 \end{pmatrix} + \begin{pmatrix} 0 & a^{11}a^{12} + a^{12}a^{22} + a^{13}a^{32} & a^{11}a^{13} + a^{12}a^{23} + a^{13}a^{33} \\ a^{21}a^{11} + a^{22}a^{21} + a^{23}a^{31} & 0 & a^{21}a^{13} + a^{22}a^{23} + a^{23}a^{33} \\ a^{31}a^{11} + a^{32}a^{21} + a^{33}a^{31} & a^{31}a^{12} + a^{32}a^{22} + a^{33}a^{32} & 0 \end{pmatrix} + \\
& \begin{pmatrix}
0 & \begin{matrix}(a^{11}a^{11} + a^{12}a^{21} + a^{13}a^{31})a^{12} + \\ (a^{11}a^{12} + a^{12}a^{22} + a^{13}a^{32})a^{22} + \\ (a^{11}a^{13} + a^{12}a^{23} + a^{13}a^{33})a^{32}\end{matrix} & \begin{matrix}(a^{11}a^{11} + a^{12}a^{21} + a^{13}a^{31})a^{13} + \\ (a^{11}a^{12} + a^{12}a^{22} + a^{13}a^{32})a^{23} + \\ (a^{11}a^{13} + a^{12}a^{23} + a^{13}a^{33})a^{33}\end{matrix} \\
\begin{matrix}(a^{21}a^{11} + a^{22}a^{21} + a^{23}a^{31})a^{11} + \\ (a^{21}a^{12} + a^{22}a^{22} + a^{23}a^{32})a^{21} + \\ (a^{21}a^{13} + a^{22}a^{23} + a^{23}a^{33})a^{31}\end{matrix} & 0 & \begin{matrix}(a^{21}a^{11} + a^{22}a^{21} + a^{23}a^{31})a^{13} + \\ (a^{21}a^{12} + a^{22}a^{22} + a^{23}a^{32})a^{23} + \\ (a^{21}a^{13} + a^{22}a^{23} + a^{23}a^{33})a^{33}\end{matrix} \\
\begin{matrix}(a^{31}a^{11} + a^{32}a^{21} + a^{33}a^{31})a^{11} + \\ (a^{31}a^{12} + a^{32}a^{22} + a^{33}a^{32})a^{21} + \\ (a^{31}a^{13} + a^{32}a^{23} + a^{33}a^{33})a^{31}\end{matrix} & \begin{matrix}(a^{31}a^{11} + a^{32}a^{21} + a^{33}a^{31})a^{12} + \\ (a^{31}a^{12} + a^{32}a^{22} + a^{33}a^{32})a^{22} + \\ (a^{31}a^{13} + a^{32}a^{23} + a^{33}a^{33})a^{32}\end{matrix} & 0
\end{pmatrix} + \dots
\end{aligned}
$$

which can be continued analogously for the higher order terms. The first line represents matrix $\bar{\mathbf{L}}$, the second and third line represents $\check{\mathbf{L}}$, and the fourth and fifth line is $\tilde{\mathbf{L}}$.

A.3 Hypothetical extraction (special case)

The special case as mentioned in Section 6 results when setting all elements a^{rc} with $r \neq c$ to zero (which is equivalent with matrix $\hat{\mathbf{A}}$. Equation (A.1) then reduces to $\bar{\mathbf{L}}$ as all other elements drop out. Thus, one considers the purely domestic value added flows only in the thus defined value chains.

B Formulation of the KWW decomposition

In this section we write the KWW approach using the terminology introduced in the main text. This results in a matrix exposition of the KWW terms, though one has to keep in mind that the KWW approach has been developed with a focus on decomposing a country's total gross exports (not the bilateral ones) as has been focused on in Section 5. Nonetheless some of the KWW terms have a bilateral counterpart, whereas some not. Therefore the matrices shown here have to be interpreted cautiously and in some cases only the row sums provide an appropriate interpretation in line of the KWW approach as will be indicated. These relations are discussed in detail in Section 5, whereas here the emphasis is on a detailed expression for the KWW terms.

B.1 KWW decomposition in matrix terms

The nine terms in the KWW decomposition can be represented using the terminology of this paper as follows:

$$\mathbf{K} = \underbrace{\hat{\mathbf{B}}\tilde{\mathbf{F}} + \tilde{\mathbf{B}}\hat{\mathbf{F}} + \widetilde{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}}_{\text{Value added exports}} + \underbrace{\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{F}}} + \widehat{\tilde{\mathbf{B}}\tilde{\mathbf{A}}\bar{\mathbf{L}}\hat{\mathbf{F}}}}_{\text{Value added re-imports}} + \underbrace{\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{A}}\bar{\mathbf{L}}\hat{\mathbf{e}}^*}}_{\text{DCdom}} + \underbrace{\widehat{(\mathbf{1}'\tilde{\mathbf{B}})}\tilde{\mathbf{F}} + \widehat{(\mathbf{1}'\tilde{\mathbf{B}})}\tilde{\mathbf{A}}\bar{\mathbf{L}}\hat{\mathbf{F}}}_{\text{Foreign content}} + \underbrace{\widehat{(\mathbf{1}'\tilde{\mathbf{B}})}\tilde{\mathbf{A}}\bar{\mathbf{L}}\hat{\mathbf{e}}^*}_{\text{DCfor}}$$

The interpretation of these terms is given as:

1. Domestic value added in direct final goods exports
2. Domestic value added in intermediate goods exports absorbed by direct importers
3. Domestic value added in intermediate goods exports re-exported to third countries
4. Domestic value added in intermediate goods exports re-imported as final goods
5. Domestic value added in intermediate goods exports re-imported as intermediate goods and finally absorbed at home
6. Double-counted intermediate exports originally produced at home
7. Foreign value added in exports of final goods
8. Foreign value added in exports of intermediate goods
9. Double-counted intermediate exports originally produced abroad

The matrices are now discussed in detail. Again it should be emphasised that the focus of the KWW approach is on a decomposition of the country's total gross exports, thus it holds that $\mathbf{K1} = \mathbf{E1}$, but not $\mathbf{K} = \mathbf{E}$ for reasons outlined in Section 5.

B.2 Matrices

B.2.1 Value added exports

KWW1: Domestic VA in direct final goods exports

$$\hat{\mathbf{B}}\tilde{\mathbf{F}} = \begin{pmatrix} b^{11} & 0 & 0 \\ 0 & b^{22} & 0 \\ 0 & 0 & b^{33} \end{pmatrix} \begin{pmatrix} 0 & f^{12} & f^{13} \\ f^{21} & 0 & f^{23} \\ f^{31} & f^{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & b^{11}f^{12} & b^{11}f^{13} \\ b^{22}f^{21} & 0 & b^{22}f^{23} \\ b^{33}f^{31} & b^{33}f^{32} & 0 \end{pmatrix}$$

KWW2: Domestic VA in intermediate goods exports absorbed by direct importer

$$\tilde{\mathbf{B}}\hat{\mathbf{F}} = \begin{pmatrix} 0 & b^{12} & b^{13} \\ b^{21} & 0 & b^{23} \\ b^{31} & b^{32} & 0 \end{pmatrix} \begin{pmatrix} f^{11} & 0 & 0 \\ 0 & f^{22} & 0 \\ 0 & 0 & f^{33} \end{pmatrix} = \begin{pmatrix} 0 & b^{12}f^{22} & b^{13}f^{33} \\ b^{21}f^{11} & 0 & b^{23}f^{33} \\ b^{31}f^{11} & b^{32}f^{22} & 0 \end{pmatrix}$$

KWW3: Domestic VA in intermediate goods exports re-exported to third countries

The third matrix is derived from

$$\begin{aligned} \tilde{\mathbf{B}}\tilde{\mathbf{F}} &= \begin{pmatrix} 0 & b^{12} & b^{13} \\ b^{21} & 0 & b^{23} \\ b^{31} & b^{32} & 0 \end{pmatrix} \begin{pmatrix} 0 & f^{12} & f^{13} \\ f^{21} & 0 & f^{23} \\ f^{31} & f^{32} & 0 \end{pmatrix} \\ &= \begin{pmatrix} b^{12}f^{21} + b^{13}f^{31} & b^{13}f^{32} & b^{12}f^{23} \\ b^{23}f^{31} & b^{21}f^{12} + b^{23}f^{32} & b^{21}f^{13} \\ b^{32}f^{21} & b^{31}f^{13} & b^{31}f^{13} + b^{32}f^{23} \end{pmatrix} \end{aligned}$$

by splitting out the diagonal elements, i.e.

$$\widetilde{\tilde{\mathbf{B}}\tilde{\mathbf{F}}} = \begin{pmatrix} 0 & b^{13}f^{32} & b^{12}f^{23} \\ b^{23}f^{31} & 0 & b^{21}f^{13} \\ b^{32}f^{21} & b^{31}f^{13} & 0 \end{pmatrix}$$

These three matrices constitute the value added exports (in a bilateral way) and are discussed in Section 4 (see equation (4.1)).

B.2.2 Re-imports of value added

KWW4: Domestic VA in intermediate goods exports re-imported as final products

The re-imports via final products are the diagonal elements of the matrix derived before

$$\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{F}}} = \begin{pmatrix} b^{12}f^{21} + b^{13}f^{31} & 0 & 0 \\ 0 & b^{21}f^{12} + b^{23}f^{32} & 0 \\ 0 & 0 & b^{31}f^{13} + b^{32}f^{23} \end{pmatrix}$$

In equation (4.1) in Section 4 this is part of the domestic consumption of value added and therefore appear on the diagonal.

KWW5: Domestic VA in intermediate goods exports re-imported as intermediate goods and finally absorbed at home

This flow can be derived from

$$\begin{aligned} \tilde{\mathbf{B}}\tilde{\mathbf{A}}\bar{\mathbf{L}}\hat{\mathbf{F}} &= \begin{pmatrix} 0 & b^{12} & b^{13} \\ b^{21} & 0 & b^{23} \\ b^{31} & b^{32} & 0 \end{pmatrix} \begin{pmatrix} 0 & a^{12} & a^{13} \\ a^{21} & 0 & a^{23} \\ a^{31} & a^{32} & 0 \end{pmatrix} \begin{pmatrix} \bar{l}^{11} & 0 & 0 \\ 0 & \bar{l}^{22} & 0 \\ 0 & 0 & \bar{l}^{33} \end{pmatrix} \begin{pmatrix} f^{11} & 0 & 0 \\ 0 & f^{22} & 0 \\ 0 & 0 & f^{33} \end{pmatrix} \\ &= \begin{pmatrix} b^{12}a^{21} + b^{13}a^{31} & b^{13}a^{32} & b^{12}a^{23} \\ b^{23}a^{31} & b^{21}a^{12} + b^{23}a^{32} & b^{21}a^{13} \\ b^{32}a^{21} & b^{31}a^{12} & b^{31}a^{13} + b^{32}a^{23} \end{pmatrix} \begin{pmatrix} \bar{l}^{11}f^{11} & 0 & 0 \\ 0 & \bar{l}^{22}f^{22} & 0 \\ 0 & 0 & \bar{l}^{33}f^{33} \end{pmatrix} \\ &= \begin{pmatrix} (b^{12}a^{21} + b^{13}a^{31})\bar{l}^{11}f^{11} & b^{13}a^{32}\bar{l}^{22}f^{22} & b^{12}a^{23}\bar{l}^{33}f^{33} \\ b^{23}a^{31}\bar{l}^{11}f^{11} & (b^{21}a^{12} + b^{23}a^{32})\bar{l}^{22}f^{22} & b^{21}a^{13}\bar{l}^{33}f^{33} \\ b^{32}a^{21}\bar{l}^{11}f^{11} & b^{31}a^{12}\bar{l}^{22}f^{22} & (b^{31}a^{13} + b^{32}a^{23})\bar{l}^{33}f^{33} \end{pmatrix} \end{aligned}$$

Here only the diagonal elements are relevant, thus

$$\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{A}}\bar{\mathbf{L}}\hat{\mathbf{F}}} = \begin{pmatrix} (b^{12}a^{21} + b^{13}a^{31})\bar{l}^{11}f^{11} & 0 & 0 \\ 0 & (b^{21}a^{12} + b^{23}a^{23})\bar{l}^{22}f^{22} & 0 \\ 0 & 0 & (b^{31}a^{13} + b^{32}a^{23})\bar{l}^{33}f^{33} \end{pmatrix}$$

This matrix has no counterpart in the SAS decomposition introduced in Section 4. The relation to the SAS gross exports decomposition is discussed in Section 5.

Note that the terms in these two matrices appear on the diagonal as they constitute domestic consumption of value added. To be aligned with the KWW decomposition of gross exports this matrices therefore has to be summed up, i.e. $\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}\mathbf{1}$ as these constitutes part of the (total) gross exports of intermediates.

B.2.3 Foreign VA in bilateral gross exports

KWW7: Foreign VA in bilateral gross exports of final goods

The foreign value added in a country's gross exports (bilateral) of final products is given by

$$(\widehat{\mathbf{1}'\tilde{\mathbf{B}}})\tilde{\mathbf{F}} = \begin{pmatrix} 0 & (b^{21}+b^{31})f^{12} & (b^{21}+b^{31})f^{13} \\ (b^{12}+b^{32})f^{21} & 0 & (b^{12}+b^{32})f^{23} \\ (b^{13}+b^{23})f^{31} & (b^{13}+b^{23})f^{32} & 0 \end{pmatrix}$$

KWW8: Foreign value added in exports of intermediate goods

Similarly, the foreign value added content in a country's gross exports (bilateral) of intermediary products is given by

$$\widehat{(\mathbf{1}'\tilde{\mathbf{B}})}\tilde{\mathbf{A}}\bar{\mathbf{L}}\hat{\mathbf{F}} = \begin{pmatrix} 0 & (b^{21}+b^{31})a^{12}\bar{l}^{22}f^{22} & (b^{21}+b^{31})a^{13}\bar{l}^{33}f^{33} \\ (b^{12}+b^{32})a^{21}\bar{l}^{11}f^{11} & 0 & (b^{12}+b^{32})a^{23}\bar{l}^{33}f^{33} \\ (b^{13}+b^{23})a^{31}\bar{l}^{11}f^{11} & (b^{13}+b^{23})a^{32}\bar{l}^{22}f^{22} & 0 \end{pmatrix}$$

Both terms appear in the "SAS-VAC approach" discussed in Section **??** and are used to discuss the relationship between this and the KWW decomposition in Section 5.

B.2.4 Double-counted terms

Finally, the KWW approach includes so-called 'double-counted' terms. These can be represented using our terminology as follows:

KWW6: Double-counted intermediate exports originally produced at home

Denote $\tilde{\mathbf{E}}\mathbf{1} = \mathbf{e}^*$. Then this term can be written as

$$\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{A}}\bar{\mathbf{L}}\hat{\mathbf{e}}^*} = \begin{pmatrix} (b^{12}a^{21} + b^{13}a^{31})\bar{l}^{11}e^{1*} & 0 & 0 \\ 0 & (b^{21}a^{12} + b^{23}a^{23})\bar{l}^{22}e^{2*} & 0 \\ 0 & 0 & (b^{31}a^{13} + b^{32}a^{23})\bar{l}^{33}e^{3*} \end{pmatrix}$$

Again, to be aligned with the KWW approach, this term has to be summed up over columns, i.e. $\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{A}}\bar{\mathbf{L}}\hat{\mathbf{e}}^*}\mathbf{1} = \tilde{\mathbf{B}}\tilde{\mathbf{A}}\bar{\mathbf{L}}\mathbf{e}^*$. In Section 5 it is shown that this term corresponds to the countries' total 'complex' exports in the SAS decomposition of gross exports.

KWW9: Double-counted intermediate exports originally produced abroad

The last double-counted term is given as

$$\widehat{(\mathbf{1}'\tilde{\mathbf{B}})}\tilde{\mathbf{A}}\bar{\mathbf{L}}\hat{\mathbf{e}}^* = \begin{pmatrix} 0 & (b^{21} + b^{31})a^{12}\bar{l}^{22}e^{2*} & (b^{21} + b^{31})a^{13}\bar{l}^{33}e^{3*} \\ (b^{12} + b^{32})a^{21}\bar{l}^{11}e^{1*} & 0 & (b^{12} + b^{32})a^{23}\bar{l}^{33}e^{3*} \\ (b^{13} + b^{23})a^{31}\bar{l}^{11}e^{1*} & (b^{13} + b^{23})a^{32}\bar{l}^{22}e^{2*} & 0 \end{pmatrix}$$

In Section 5 we show that in the SAS decomposition of gross exports this corresponds to the bilateral foreign content of intermediary exports which are finally assembled and absorbed in the latter country.

It is important to note, that this is a bilateral representation of the KWW approach, whereas this has been genuinely developed for a country's total exports. // REFERENCE TO bilateral paper // Thus, strictly speaking, the terms should be post-multiplied by a vector of ones (i.e. building the row sums) which are indicated in the columns denote 'RowSums'. However, as this is useful in the discussion and point of reference below this is not done here.

Table B.1: KWW decomposition (9 terms)

	Gross output					Value added				
	EU-28	China	USA	RoW	Total	EU-28	China	USA	RoW	Total
KWW1: Domestic GO / VA in direct final goods exports										
EU-28	2,333	232	348	1,831	4,744	932	98	146	771	1,947
China	557	0	672	2,421	3,650	151	0	180	690	1,020
USA	230	86	0	891	1,207	114	38	0	433	586
RoW	1,009	561	1,102	2,330	5,001	385	204	412	913	1,914
Total	4,129	879	2,121	7,473	14,603	1,582	340	738	2,807	5,467
KWW2: Domestic GO / VA in int. exports absorbed by direct importers										
EU-28	2,298	319	549	2,261	5,427	952	133	231	957	2,272
China	392	0	439	1,878	2,708	110	0	116	557	783
USA	328	147	0	1,169	1,644	170	71	0	573	814
RoW	1,547	1,899	1,748	4,312	9,507	660	723	732	1,810	3,926
Total	4,565	2,365	2,736	9,620	19,285	1,893	927	1,079	3,896	7,795
KWW3: Domestic GO / VA in int. exports re-exported to third countries										
EU-28	621	96	166	626	1,509	248	39	68	256	610
China	183	0	111	263	557	51	0	31	72	154
USA	119	25	0	166	310	61	13	0	83	157
RoW	584	143	315	661	1,703	233	56	119	263	671
Total	1,507	265	592	1,715	4,079	593	107	217	674	1,592
KWW4: Domestic VA / GO in int. exports re-imported as final goods										
EU-28	99	0	0	0	99	41	0	0	0	41
China	0	76	0	0	76	0	21	0	0	21
USA	0	0	134	0	134	0	0	61	0	61
RoW	0	0	0	336	336	0	0	0	125	125
Total	99	76	134	336	644	41	21	61	125	248
KWW5: Domestic VA / GO in int. exports re-imported as int. inputs ...										
EU-28	72	0	0	0	72	30	0	0	0	30
China	0	137	0	0	137	0	38	0	0	38
USA	0	0	128	0	128	0	0	61	0	61
RoW	0	0	0	355	355	0	0	0	140	140
Total	72	137	128	355	693	30	38	61	140	270
KWW6: Double-counted inter. exports originally produced at home										
EU-28	104	0	0	0	104	42	0	0	0	42
China	0	84	0	0	84	0	23	0	0	23
USA	0	0	28	0	28	0	0	13	0	13
RoW	0	0	0	204	204	0	0	0	77	77
Total	104	84	28	204	420	42	23	13	77	156
KWW7: Foreign GO / VA in exports of final goods										
EU-28	1,142	104	151	796	2,193	454	40	60	321	875
China	82	0	102	341	525	30	0	37	126	193
USA	41	20	0	152	213	15	7	0	58	81
RoW	341	217	473	763	1,793	135	81	181	294	691
Total	1,606	341	726	2,051	4,724	634	128	279	799	1,840
KWW8: Foreign GO / VA in exports of intermediary goods										
EU-28	872	79	168	707	1,827	355	31	69	295	749
China	38	0	52	210	300	14	0	19	81	115
USA	34	16	0	174	224	13	6	0	70	90
RoW	301	539	497	1,053	2,390	123	210	198	423	955
Total	1,245	634	718	2,146	4,742	505	248	287	868	1,908
KWW9: Double-counted int. exports orginally produced abroad										
EU-28	1,182	26	37	384	1,628	472	10	15	159	656
China	45	0	12	152	210	17	0	4	57	78
USA	48	5	0	112	164	19	2	0	43	64
RoW	369	286	112	672	1,439	148	108	44	264	564
Total	1,643	318	161	1,320	3,442	655	120	64	523	1,361

C The property of inverse matrices

An important ingredient to show the relationships between these two decomposition approaches is the 'property of inverse matrices'. This states that $\mathbf{L}\mathbf{L}^{-1} = \mathbf{L}(\mathbf{I}-\mathbf{A}) = \mathbf{I}$ and $\mathbf{L}^{-1}\mathbf{L} = (\mathbf{I}-\mathbf{A})\mathbf{L} = \mathbf{I}$ from which it follows that $\mathbf{L}(\mathbf{I}-\mathbf{A}) = (\mathbf{I}-\mathbf{A})\mathbf{L}$. This, of course, also holds when considering the diagonal elements only, i.e. $\bar{\mathbf{L}}\bar{\mathbf{L}}^{-1} = \bar{\mathbf{L}}(\mathbf{I}-\hat{\mathbf{A}}) = \mathbf{I}$ and $\bar{\mathbf{L}}^{-1}\bar{\mathbf{L}} = (\mathbf{I}-\hat{\mathbf{A}})\bar{\mathbf{L}} = \mathbf{I}$ from which it follows that $\bar{\mathbf{L}}(\mathbf{I}-\hat{\mathbf{A}}) = (\mathbf{I}-\hat{\mathbf{A}})\bar{\mathbf{L}}$. It further follows that $\bar{\mathbf{L}}-\mathbf{I} = \bar{\mathbf{L}}\hat{\mathbf{A}}$ and $\bar{\mathbf{L}}-\mathbf{I} = \hat{\mathbf{A}}\bar{\mathbf{L}}$. Using these identities one can show that

$$\mathbf{L} = \mathbf{L}\tilde{\mathbf{A}}\bar{\mathbf{L}} + \bar{\mathbf{L}} = \bar{\mathbf{L}}\tilde{\mathbf{A}}\mathbf{L} + \bar{\mathbf{L}}$$

The off-diagonal blocks are then given by $\tilde{\mathbf{L}} = (\widetilde{\mathbf{L}\tilde{\mathbf{A}}\bar{\mathbf{L}}}) = (\widetilde{\bar{\mathbf{L}}\tilde{\mathbf{A}}\mathbf{L}})$. The diagonal elements are given by $\hat{\mathbf{L}} = (\widehat{\mathbf{L}\tilde{\mathbf{A}}\bar{\mathbf{L}}}) + \bar{\mathbf{L}} = (\widehat{\bar{\mathbf{L}}\tilde{\mathbf{A}}\mathbf{L}}) + \bar{\mathbf{L}}$ or $\hat{\mathbf{L}} = (\widehat{\tilde{\mathbf{L}}\tilde{\mathbf{A}}\bar{\mathbf{L}}}) + \bar{\mathbf{L}} = (\widehat{\bar{\mathbf{L}}\tilde{\mathbf{A}}\tilde{\mathbf{L}}}) + \bar{\mathbf{L}}$

PROOF: This can be shown by starting from above properties $\mathbf{I} = \mathbf{L}(\mathbf{I}-\mathbf{A}) = \mathbf{L} - \mathbf{L}\mathbf{A}$. Post-multiplying with $-\bar{\mathbf{L}}$ results in $-\bar{\mathbf{L}} = \mathbf{L}\mathbf{A}\bar{\mathbf{L}} - \mathbf{L}\bar{\mathbf{L}}$. Adding $\mathbf{L}$ on both sides and re-arranging leads to

$$\mathbf{L} - \bar{\mathbf{L}} = \mathbf{L}\mathbf{A}\bar{\mathbf{L}} - \mathbf{L}\bar{\mathbf{L}} + \mathbf{L} = \mathbf{L}\mathbf{A}\bar{\mathbf{L}} - \mathbf{L}(\bar{\mathbf{L}} - \mathbf{I}) = \mathbf{L}\mathbf{A}\bar{\mathbf{L}} - \mathbf{L}\hat{\mathbf{A}}\bar{\mathbf{L}} = \mathbf{L}(\mathbf{A} - \hat{\mathbf{A}})\bar{\mathbf{L}} = \mathbf{L}\tilde{\mathbf{A}}\bar{\mathbf{L}}$$

which results in above formula $\mathbf{L} = \mathbf{L}\tilde{\mathbf{A}}\bar{\mathbf{L}} + \bar{\mathbf{L}}$. Analogously, the second statement can be proved.

The formula for the off-diagonal elements results from $\tilde{\bar{\mathbf{L}}} = \mathbf{0}$. The first formula for the diagonal elements are clear by definition. The second formula follows from

$$(\widehat{\tilde{\mathbf{L}}\tilde{\mathbf{A}}\bar{\mathbf{L}}}) = \big((\widehat{\mathbf{L} - \hat{\mathbf{L}})\tilde{\mathbf{A}}\bar{\mathbf{L}}}\big) = (\widehat{\mathbf{L}\tilde{\mathbf{A}}\bar{\mathbf{L}}}) - (\widehat{\hat{\mathbf{L}}\tilde{\mathbf{A}}\bar{\mathbf{L}}}) = (\widehat{\mathbf{L}\tilde{\mathbf{A}}\bar{\mathbf{L}}})$$

as $(\widehat{\hat{\mathbf{L}}\tilde{\mathbf{A}}\bar{\mathbf{L}}}) = \mathbf{0}$. The reason for this is that when pre- and post-multiplying an off-diagonal matrix (with only 0's at the diagonal) with diagonal matrices the 0's at the diagonal remain. Therefore taking only the diagonal elements and setting the off-diagonal elements to 0's results in a matrix $\mathbf{0}$. $\square$

D Hypothetical extraction method

D.1 Multiplier decomposition using hypothetical extraction

In Section 6 an example of using the hypothetical extraction method has been discussed. Here we show the various matrices appearing in that context using the example above to allow a better understanding of the hypothetical extraction method. For illustrative purposes let countries 1 and 2 be the 'EU countries' (following the example in Section 6). In this case we have

$$\ddot{\mathbf{A}} = \left(\begin{array}{cc|c} a^{11} & a^{12} & 0 \\ a^{21} & a^{22} & 0 \\ \hline 0 & 0 & a^{33} \end{array}\right)$$

thus a^{13}, a^{23}, a^{31}, and a^{32} are set to zero. In Appendix Section A it was further argued that the power expansion of the Leontief multiplier matrices can be split into the three terms$\mathbf{L} = \bar{\mathbf{L}} + \breve{\mathbf{L}} + \tilde{\mathbf{L}}$. Using the hypothetical extraction method allows to split these matrices into the value chains considered. In the concrete example above, therefore the splits are as follows (the lines distinguish countries 1 and 2 from 3):

$$\bar{\mathbf{L}} = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) + \left(\begin{array}{cc|c} a^{11} & 0 & 0 \\ 0 & a^{22} & 0 \\ \hline 0 & 0 & a^{33} \end{array}\right) + \left(\begin{array}{cc|c} a^{11}a^{11} & 0 & 0 \\ 0 & a^{22}a^{22} & 0 \\ \hline 0 & 0 & a^{33}a^{33} \end{array}\right) + \left(\begin{array}{cc|c} a^{11}a^{11}a^{11} & 0 & 0 \\ 0 & a^{22}a^{22}a^{22} & 0 \\ \hline 0 & 0 & a^{33}a^{33}a^{33} \end{array}\right) + \cdots +$$

Thus, nothing changes when considering the pure intra-country flows. This already changes for the domestic flows with multiple border crossings which can be split into the ones within countries 1 and 2 and the others. Matrix $\breve{\mathbf{L}}$ is therefore split as follows:

$$\begin{aligned}
\breve{\mathbf{L}} \quad = \quad & \left(\begin{array}{cc|c} a^{12}a^{21}+0 & 0 & 0 \\ 0 & a^{21}a^{12}+0 & 0 \\ \hline 0 & 0 & 0+0 \end{array}\right) + \\
& \left(\begin{array}{cc|c} \begin{array}{c}(a^{12}a^{21}+0)a^{11}+ \\ (a^{11}a^{12}+a^{12}a^{22}+0)a^{21}+ \\ (0+0+0)0\end{array} & 0 & 0 \\ 0 & \begin{array}{c}(a^{21}a^{11}+a^{22}a^{21}+0)a^{12}+ \\ (a^{21}a^{12}+0)a^{22}+ \\ (0+0+0)0\end{array} & 0 \\ \hline 0 & 0 & \begin{array}{c}(0+0+0)0+ \\ (0+0+0)0+ \\ (0+0+0)0\end{array} \end{array}\right) + \ldots \\
+ \quad & \left(\begin{array}{cc|c} 0+a^{13}a^{31} & 0 & 0 \\ 0 & 0+a^{23}a^{32} & 0 \\ \hline 0 & 0 & a^{31}a^{13}+a^{32}a^{23} \end{array}\right) + \\
& \left(\begin{array}{cc|c} \begin{array}{c}(0+a^{13}a^{31})a^{11}+ \\ (0+0+a^{13}a^{32})a^{21}+ \\ (a^{11}a^{13}+a^{12}a^{23}+a^{13}a^{33})a^{31}\end{array} & 0 & 0 \\ 0 & \begin{array}{c}(0+a^{22}a^{21}+a^{23}a^{31})a^{12}+ \\ (0+a^{23}a^{32})a^{22}+ \\ (a^{21}a^{13}+a^{22}a^{23}+a^{23}a^{33})a^{32}\end{array} & 0 \\ \hline 0 & 0 & \begin{array}{c}(a^{31}a^{11}+a^{32}a^{21}+a^{33}a^{31})a^{13}+ \\ (a^{31}a^{12}+a^{32}a^{22}+a^{33}a^{32})a^{23}+ \\ (a^{31}a^{13}+a^{32}a^{23}+)a^{33}\end{array} \end{array}\right) + \ldots
\end{aligned}$$

In the first two lines all flows including flows of countries 1 and 2 with 3 are nullified as a^{13}, a^{23}, a^{31}, and a^{32} are set to zero (which is indicated with the 0s). These extracted flows are however separately traced in the last two lines of the equation. Note that these also include flows from countries 1 and 2 to 3 which are not purely internal. This is similarly the case for the off-diagonal blocks represented in the next equation.

$$
\begin{aligned}
\tilde{\mathbf{L}} \quad = \quad & \left(\begin{array}{cc|c} 0 & a^{12} & 0 \\ a^{21} & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) + \left(\begin{array}{cc|c} 0 & a^{11}a^{12}+a^{12}a^{22}+0 & 0+0+0 \\ a^{21}a^{11}+a^{22}a^{21}+0 & 0 & 0+0+0 \\ \hline 0+0+0 & 0+0+0 & 0 \end{array}\right) + \\
& \left(\begin{array}{cc|c}
0 & \begin{array}{c}(a^{11}a^{11}+a^{12}a^{21}+0)a^{12}+\\(a^{11}a^{12}+a^{12}a^{22}+0)a^{22}+\\(0+0+0)0\end{array} & \begin{array}{c}(a^{11}a^{11}+a^{12}a^{21}+0)0+\\(a^{11}a^{12}+a^{12}a^{22}+0)0+\\(0+0+0)a^{33}\end{array} \\
\begin{array}{c}(a^{21}a^{11}+a^{22}a^{21}+0)a^{11}+\\(a^{21}a^{12}+a^{22}a^{22}+0)a^{21}+\\(0+0+0)0\end{array} & 0 & \begin{array}{c}(a^{21}a^{11}+a^{22}a^{21}+0)0+\\(a^{21}a^{12}+a^{22}a^{22}+0)0+\\(0+0+0)a^{33}\end{array} \\ \hline
\begin{array}{c}(0+0+0)a^{11}+\\(0+0+0)a^{21}+\\(0+0+a^{33}a^{33})0\end{array} & \begin{array}{c}(0+0+0)a^{12}+\\(0+0+0)a^{22}+\\(0+0+a^{33}a^{33})0\end{array} & 0
\end{array}\right) + \dots \\
+ \; & \left(\begin{array}{cc|c} 0 & 0 & a^{13} \\ 0 & 0 & a^{23} \\ \hline a^{31} & a^{32} & 0 \end{array}\right) + \left(\begin{array}{cc|c} 0 & 0+0+a^{13}a^{32} & a^{11}a^{13}+a^{12}a^{23}+a^{13}a^{33} \\ 0+0+a^{23}a^{31} & 0 & a^{21}a^{13}+a^{22}a^{23}+a^{23}a^{33} \\ \hline a^{31}a^{11}+a^{32}a^{21}+a^{33}a^{31} & a^{31}a^{12}+a^{32}a^{22}+a^{33}a^{32} & 0 \end{array}\right) + \\
& \left(\begin{array}{cc|c}
0 & \begin{array}{c}(0+0+a^{13}a^{31})a^{12}+\\(0+0+a^{13}a^{32})a^{22}+\\(a^{11}a^{13}+a^{12}a^{23}+a^{13}a^{33})a^{32}\end{array} & \begin{array}{c}(0+0+a^{13}a^{31})a^{13}+\\(0+0+a^{13}a^{32})a^{23}+\\(a^{11}a^{13}+a^{12}a^{23}+a^{13}a^{33})a^{33}\end{array} \\
\begin{array}{c}(0+0+a^{23}a^{31})a^{11}+\\(0+0+a^{23}a^{32})a^{21}+\\(a^{21}a^{13}+a^{22}a^{23}+a^{23}a^{33})a^{31}\end{array} & 0 & \begin{array}{c}(0+0+a^{23}a^{31})a^{13}+\\(0+0+a^{23}a^{32})a^{23}+\\(a^{21}a^{13}+a^{22}a^{23}+a^{23}a^{33})a^{33}\end{array} \\ \hline
\begin{array}{c}(a^{31}a^{11}+a^{32}a^{21}+a^{33}a^{31})a^{11}+\\(a^{31}a^{12}+a^{32}a^{22}+a^{33}a^{32})a^{21}+\\(a^{31}a^{13}+a^{32}a^{23}+a^{33}a^{33})a^{31}\end{array} & \begin{array}{c}(a^{31}a^{11}+a^{32}a^{21}+a^{33}a^{31})a^{12}+\\(a^{31}a^{12}+a^{32}a^{22}+a^{33}a^{32})a^{22}+\\(a^{31}a^{13}+a^{32}a^{23}+a^{33}a^{33})a^{32}\end{array} & 0
\end{array}\right) + \dots
\end{aligned}
$$

D.2 Derivation of decomposition

In this section the derivation of equation (6.1) is presented. The starting point is equation (4.1) which is reproduced here.

$$\mathbf{T} = \mathbf{B}\mathbf{F} = \underbrace{\overbrace{(\bar{\mathbf{B}}\hat{\mathbf{F}} + \breve{\mathbf{B}}\hat{\mathbf{F}})}^{=\hat{\mathbf{B}}\hat{\mathbf{F}}} + \widehat{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}}_{\text{Domestic consumption}} + \underbrace{\overbrace{(\bar{\mathbf{B}}\tilde{\mathbf{F}} + \breve{\mathbf{B}}\tilde{\mathbf{F}})}^{=\hat{\mathbf{B}}\tilde{\mathbf{F}}} + \tilde{\mathbf{B}}\hat{\mathbf{F}} + \widetilde{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}}_{\text{Value added exports}}$$

Inserting for $\mathbf{B}$ from equation (6.1) and using $\mathbf{F} = \hat{\mathbf{F}} + \tilde{\mathbf{F}}$ results in

$$
\begin{aligned}
\mathbf{T} \quad &= \quad \Big(\bar{\mathbf{B}} + \underbrace{(\hat{\ddot{\mathbf{B}}} - \bar{\mathbf{B}}) + (\hat{\mathbf{B}} - \hat{\ddot{\mathbf{B}}})}_{\breve{\mathbf{B}}} + \underbrace{\tilde{\ddot{\mathbf{B}}} + (\tilde{\mathbf{B}} - \tilde{\ddot{\mathbf{B}}})}_{\tilde{\mathbf{B}}}\Big)\big(\hat{\mathbf{F}} + \tilde{\mathbf{F}}\big) \\
&= \quad \Big(\bar{\mathbf{B}} + \underbrace{(\hat{\ddot{\mathbf{B}}} - \bar{\mathbf{B}}) + (\hat{\mathbf{B}} - \hat{\ddot{\mathbf{B}}})}_{\breve{\mathbf{B}}} + \underbrace{\tilde{\ddot{\mathbf{B}}} + (\tilde{\mathbf{B}} - \tilde{\ddot{\mathbf{B}}})}_{\tilde{\mathbf{B}}}\Big)\hat{\mathbf{F}} + \Big(\bar{\mathbf{B}} + \underbrace{(\hat{\ddot{\mathbf{B}}} - \bar{\mathbf{B}}) + (\hat{\mathbf{B}} - \hat{\ddot{\mathbf{B}}})}_{\breve{\mathbf{B}}} + \underbrace{\tilde{\ddot{\mathbf{B}}} + (\tilde{\mathbf{B}} - \tilde{\ddot{\mathbf{B}}})}_{\tilde{\mathbf{B}}}\Big)\tilde{\mathbf{F}}
\end{aligned}
$$

Re-arranging the terms

$$
\begin{aligned}
\mathbf{T} \quad = \quad & \bar{\mathbf{B}}\hat{\mathbf{F}} + \underbrace{(\hat{\ddot{\mathbf{B}}} - \bar{\mathbf{B}})\hat{\mathbf{F}} + (\hat{\mathbf{B}} - \hat{\ddot{\mathbf{B}}})\hat{\mathbf{F}}}_{\breve{\mathbf{B}}\hat{\mathbf{F}}} + \underbrace{\tilde{\ddot{\mathbf{B}}}\hat{\mathbf{F}} + (\tilde{\mathbf{B}} - \tilde{\ddot{\mathbf{B}}})\hat{\mathbf{F}}}_{\tilde{\mathbf{B}}\hat{\mathbf{F}}} + \\
& \bar{\mathbf{B}}\tilde{\mathbf{F}} + \underbrace{(\hat{\ddot{\mathbf{B}}} - \bar{\mathbf{B}})\tilde{\mathbf{F}} + (\hat{\mathbf{B}} - \hat{\ddot{\mathbf{B}}})\tilde{\mathbf{F}}}_{\breve{\mathbf{B}}\tilde{\mathbf{F}}} + \underbrace{\tilde{\ddot{\mathbf{B}}}\tilde{\mathbf{F}} + (\tilde{\mathbf{B}} - \tilde{\ddot{\mathbf{B}}})\tilde{\mathbf{F}}}_{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}
\end{aligned}
$$

and splitting the last term into the diagonal and the off-diagonal blocks provides

$$
\begin{aligned}
\mathbf{T} \quad = \quad & \bar{\mathbf{B}}\hat{\mathbf{F}} + \underbrace{(\hat{\ddot{\mathbf{B}}}\hat{\mathbf{F}} - \bar{\mathbf{B}}\hat{\mathbf{F}}) + (\hat{\mathbf{B}}\hat{\mathbf{F}} - \hat{\ddot{\mathbf{B}}}\hat{\mathbf{F}})}_{\breve{\mathbf{B}}\hat{\mathbf{F}}} + \underbrace{\tilde{\ddot{\mathbf{B}}}\hat{\mathbf{F}} + (\tilde{\mathbf{B}}\hat{\mathbf{F}} - \tilde{\ddot{\mathbf{B}}}\hat{\mathbf{F}})}_{\tilde{\mathbf{B}}\hat{\mathbf{F}}} + \\
& \bar{\mathbf{B}}\tilde{\mathbf{F}} + \underbrace{(\hat{\ddot{\mathbf{B}}}\tilde{\mathbf{F}} - \bar{\mathbf{B}}\tilde{\mathbf{F}}) + (\hat{\mathbf{B}}\tilde{\mathbf{F}} - \hat{\ddot{\mathbf{B}}}\tilde{\mathbf{F}})}_{\breve{\mathbf{B}}\tilde{\mathbf{F}}} + \underbrace{\widehat{\tilde{\ddot{\mathbf{B}}}\tilde{\mathbf{F}}} + (\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{F}} - \tilde{\ddot{\mathbf{B}}}\tilde{\mathbf{F}}}) + \widetilde{\tilde{\ddot{\mathbf{B}}}\tilde{\mathbf{F}}} + (\widetilde{\tilde{\mathbf{B}}\tilde{\mathbf{F}} - \tilde{\ddot{\mathbf{B}}}\tilde{\mathbf{F}}})}_{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}
\end{aligned}
$$

As explained in Section 4 the diagonal elements constitute the re-imports of value added and are therefore included in the domestic consumption part of equation (6.2) reproduced here for simplicity:

$$
\begin{aligned}
\mathbf{T} \quad = \quad & \underbrace{\overbrace{\bar{\mathbf{B}}\hat{\mathbf{F}} + \underbrace{(\hat{\ddot{\mathbf{B}}} - \bar{\mathbf{B}})\hat{\mathbf{F}} + (\hat{\mathbf{B}} - \hat{\ddot{\mathbf{B}}})\hat{\mathbf{F}}}_{\breve{\mathbf{B}}\hat{\mathbf{F}}}}^{\hat{\mathbf{B}}\hat{\mathbf{F}}} + \underbrace{\widehat{\tilde{\ddot{\mathbf{B}}}\tilde{\mathbf{F}}} + \widehat{(\tilde{\mathbf{B}} - \tilde{\ddot{\mathbf{B}}})\tilde{\mathbf{F}}}}_{\widehat{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}}}_{\text{Domestic consumption}} + \\
& \underbrace{\overbrace{\bar{\mathbf{B}}\tilde{\mathbf{F}} + \underbrace{(\hat{\ddot{\mathbf{B}}} - \bar{\mathbf{B}})\tilde{\mathbf{F}} + (\hat{\mathbf{B}} - \hat{\ddot{\mathbf{B}}})\tilde{\mathbf{F}}}_{\breve{\mathbf{B}}\tilde{\mathbf{F}}}}^{\hat{\mathbf{B}}\tilde{\mathbf{F}}} + \underbrace{\tilde{\ddot{\mathbf{B}}}\hat{\mathbf{F}} + (\tilde{\mathbf{B}} - \tilde{\ddot{\mathbf{B}}})\hat{\mathbf{F}}}_{\tilde{\mathbf{B}}\hat{\mathbf{F}}} + \underbrace{\widetilde{\tilde{\ddot{\mathbf{B}}}\tilde{\mathbf{F}}} + \widetilde{(\tilde{\mathbf{B}} - \tilde{\ddot{\mathbf{B}}})\tilde{\mathbf{F}}}}_{\widetilde{\tilde{\mathbf{B}}\tilde{\mathbf{F}}}}}_{\text{Value added exports}}
\end{aligned}
$$

As outlined above, the first five terms are domestic consumption of value added (including re-imports), and the remaining terms constitute value added exports.

D.3 Detailed outline

It is enlightening to look at these terms in detail. We split according to domestic and foreign absorption of value added. In the simplified example outlined here let again countries 1 and 2 constitute the 'EU countries' following the example given in Appendix Section A.

D.3.1 Domestic consumption

Writing them following the expressions provided in Appendix Section A[56] and in gross output terms (i.e. not pre-multiplied with the value added coefficients vector), the first three terms are just split are equal

[56] Only the first and second terms of the power expansion are presented.

to those already presented in Section 4 (here in terms of gross output):

$$\bar{\mathbf{L}}\hat{\mathbf{F}} = \left(\begin{array}{cc|c} (1+a^{11}+a^{11}a^{11}+\dots)f^{11} & 0 & 0 \\ 0 & (1+a^{22}+a^{22}a^{22}+\dots)f^{22}+\dots & 0 \\ \hline 0 & 0 & (1+a^{33}+a^{33}a^{33}+\dots)f^{33}+\dots \end{array}\right)$$

$$(\hat{\hat{\mathbf{L}}}-\bar{\mathbf{L}})\hat{\mathbf{F}} = \left(\begin{array}{cc|c} (a^{12}a^{21}+\dots)f^{11} & 0 & 0 \\ 0 & (a^{21}a^{12}+\dots)f^{22} & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

$$(\hat{\mathbf{L}}-\hat{\hat{\mathbf{L}}})\hat{\mathbf{F}} = \left(\begin{array}{cc|c} (a^{13}a^{31}+\dots)f^{11} & 0 & 0 \\ 0 & (a^{23}a^{32}+\dots)f^{22} & 0 \\ \hline 0 & 0 & (a^{31}a^{13}+a^{32}a^{23}+\dots)f^{33} \end{array}\right)$$

The horizontal and vertical lines distinguish between countries 1 and 2, and 3.[57] The next matrix results from

$$\tilde{\tilde{\mathbf{L}}}\tilde{\mathbf{F}} = \begin{pmatrix} 0 & a^{11}a^{12}+a^{12}a^{22}+\dots & 0 \\ a^{21}a^{11}+a^{22}a^{21}+\dots & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & f^{12} & f^{13} \\ f^{21} & 0 & f^{23} \\ f^{31} & f^{32} & 0 \end{pmatrix} =$$

$$\begin{pmatrix} (a^{11}a^{12}+a^{12}a^{22}+\dots)f^{21} & 0 & (a^{11}a^{12}+a^{12}a^{22}+\dots)f^{23} \\ 0 & (a^{21}a^{11}+a^{22}a^{21}+\dots)f^{12} & (a^{21}a^{11}+a^{22}a^{21}+\dots)f^{13} \\ 0 & 0 & 0 \end{pmatrix}$$

From this only the diagonal elements are taken constituting the re-imports only including chains involving countries 1 an 2, i.e.

$$\widehat{\tilde{\tilde{\mathbf{L}}}\tilde{\mathbf{F}}} = \left(\begin{array}{cc|c} (a^{11}a^{12}+a^{12}a^{22}+\dots)f^{21} & 0 & 0 \\ 0 & (a^{21}a^{11}+a^{22}a^{21}+\dots)f^{12} & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

The remaining flows are given by $(\tilde{\mathbf{L}}-\tilde{\tilde{\mathbf{L}}})\tilde{\mathbf{F}}$ resulting in

$$\begin{pmatrix} \begin{matrix}(a^{13}a^{32}+\dots)f^{21}+\\(a^{11}a^{13}+a^{12}a^{23}+a^{13}a^{33}+\dots)f^{31}\end{matrix} & (a^{11}a^{13}+a^{12}a^{23}+a^{13}a^{33}+\dots)f^{32} & (a^{13}a^{32}+\dots)f^{23} \\ (a^{21}a^{13}+a^{22}a^{23}+a^{23}a^{33}+\dots)f^{31} & \begin{matrix}(a^{23}a^{31}+\dots)f^{12}+\\(a^{21}a^{13}+a^{22}a^{23}+a^{23}a^{33}+\dots)f^{32}\end{matrix} & (a^{23}a^{31}+\dots)f^{13} \\ (a^{31}a^{12}+a^{32}a^{22}+a^{33}a^{32}+\dots)f^{21} & (a^{31}a^{11}+a^{32}a^{21}+a^{33}a^{31}+\dots)f^{12} & \begin{matrix}(a^{31}a^{11}+a^{32}a^{21}+a^{33}a^{31}+\dots)f^{13}+\\(a^{31}a^{12}+a^{32}a^{22}+a^{33}a^{32}+\dots)f^{23}\end{matrix} \end{pmatrix}$$

[57]It should be emphasised that the results do not depend on the ordering of countries.

from which only the diagonal elements are taken, i.e.

$$(\widehat{\tilde{\mathbf{L}}-\tilde{\ddot{\mathbf{L}}}})\tilde{\mathbf{F}} = \left(\begin{array}{cc|c} \begin{matrix}(a^{13}a^{32}+\ldots)f^{21}+\\(a^{11}a^{13}+a^{12}a^{23}+a^{13}a^{33}+\ldots)f^{31}\end{matrix} & 0 & 0 \\ 0 & \begin{matrix}(a^{23}a^{31}+\ldots)f^{12}+\\(a^{21}a^{13}+a^{22}a^{23}+a^{23}a^{33}+\ldots)f^{32}\end{matrix} & 0 \\ \hline 0 & 0 & \begin{matrix}(a^{31}a^{11}+a^{32}a^{21}+a^{33}a^{31}+\ldots)f^{13}+\\(a^{31}a^{12}+a^{32}a^{22}+a^{33}a^{32}+\ldots)f^{23}\end{matrix} \end{array}\right)$$

which again constitute re-imports of value added, however now involving all countries along the value chains, but excluding the inter-linkages between countries 1 and 2.

D.3.2 Value added exports

The first term for the value added exports is given by

$$\bar{\mathbf{L}}\tilde{\mathbf{F}} = \left(\begin{array}{cc|c} 0 & (a^{11}a^{11}+\ldots)f^{12} & (a^{11}a^{11}+\ldots)f^{13} \\ (a^{22}a^{22}+\ldots)f^{21} & 0 & (a^{22}a^{22}+\ldots)f^{23} \\ \hline (a^{33}a^{33}+\ldots)f^{31} & (a^{33}a^{33}+\ldots)f^{32} & 0 \end{array}\right)$$

which captures the purely domestic flows. Final products are assembled in the same country and finally exported. The second term captures the international production linkages between countries 1 and 2 with the finally assembled in these countries and being product being exported (also including country 3 as export destination).

$$(\hat{\ddot{\mathbf{L}}}-\bar{\mathbf{L}})\tilde{\mathbf{F}} = \left(\begin{array}{cc|c} 0 & (a^{12}a^{21}+\ldots)f^{12} & (a^{12}a^{21}+\ldots)f^{13} \\ (a^{21}a^{12}+\ldots)f^{21} & 0 & (a^{21}a^{12}+\ldots)f^{23} \\ \hline 0 & 0 & 0 \end{array}\right)$$

The third term captures all remaining flows, i.e. the remaining chains including country 3.

$$(\hat{\mathbf{L}}-\hat{\ddot{\mathbf{L}}})\tilde{\mathbf{F}} = \left(\begin{array}{cc|c} 0 & (a^{13}a^{31}+\ldots)f^{12} & (a^{13}a^{31}+\ldots)f^{13} \\ (a^{23}a^{32}+\ldots)f^{21} & 0 & (a^{23}a^{32}+\ldots)f^{23} \\ \hline (a^{31}a^{13}+a^{32}a^{23}+\ldots)f^{31} & (a^{31}a^{13}+a^{32}a^{23}+\ldots)f^{32} & 0 \end{array}\right)$$

The next term includes again only the flows between countries 1 and 2, however for chains in which products are finally assembled in the country of absorption (thus involving only countries 1 and 2),

$$\hat{\ddot{\mathbf{L}}}\hat{\mathbf{F}} = \left(\begin{array}{cc|c} 0 & (a^{11}a^{12}+a^{12}a^{22}+\ldots)f^{22} & 0 \\ (a^{21}a^{11}+a^{22}a^{21}+\ldots)f^{11} & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

with the next term capturing the remaining more complex flows including country 3:

$$(\tilde{\mathbf{L}}-\tilde{\ddot{\mathbf{L}}})\hat{\mathbf{F}} = \left(\begin{array}{cc|c} 0 & (a^{13}a^{32}+\ldots)f^{22} & (a^{11}a^{13}+a^{12}a^{23}+a^{13}a^{33}+\ldots)f^{33} \\ (a^{23}a^{31}+\ldots)f^{11} & 0 & (a^{21}a^{13}+a^{22}a^{23}+a^{23}a^{33}+\ldots)f^{33} \\ \hline (a^{31}a^{11}+a^{32}a^{21}+a^{33}a^{31}+\ldots)f^{11} & (a^{31}a^{12}+a^{32}a^{22}+a^{33}a^{32}+\ldots)f^{22} & 0 \end{array}\right)$$

The last but one term captures the chains between countries 1 and 2 of products which are finally assembled in one of these countries and then shipped as a final product to country 3. This consists of the off-diagonal elements of the matrix already shown above:

$$\widetilde{\tilde{\tilde{\mathbf{L}}}\tilde{\mathbf{F}}} = \left(\begin{array}{cc|c} 0 & 0 & (a^{11}a^{12} + a^{12}a^{22} + \dots)f^{23} \\ 0 & 0 & (a^{21}a^{11} + a^{22}a^{21} + \dots)f^{13} \\ \hline 0 & 0 & 0 \end{array} \right)$$

Finally, the last term captures the remaining chains including all countries, i.e. particularly including country 3. Again, here the off-diagonal elements of the matrix shown above are included, resulting in

$$(\widetilde{\tilde{\mathbf{L}} - \tilde{\tilde{\mathbf{L}}}})\tilde{\mathbf{F}} = \left(\begin{array}{cc|c} 0 & (a^{11}a^{13} + a^{12}a^{23} + a^{13}a^{33} + \dots)f^{32} & (a^{13}a^{32} + \dots)f^{23} \\ (a^{21}a^{13} + a^{22}a^{23} + a^{23}a^{33} + \dots)f^{31} & 0 & (a^{23}a^{31} + \dots)f^{13} \\ \hline (a^{31}a^{12} + a^{32}a^{22} + a^{33}a^{32} + \dots)f^{21} & (a^{31}a^{11} + a^{32}a^{21} + a^{33}a^{31} + \dots)f^{12} & 0 \end{array} \right)$$

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